Quelques exercices supplémentaires

**Exercice 15.1.** Let $M$ be an oriented manifold with boundary. Show that $M$ has a natural orientation.

**Exercice 15.2.** Show that for $k$-form $\alpha$ and $n - k - 1$ form $\beta$

$$
\int_M \alpha \wedge d\beta = (-1)^k \left[ \int_M \alpha \wedge \beta - \int_M d\alpha \wedge \beta \right].
$$

This is the *integration by parts* formula for differential forms.

**Exercice 15.3.** Consider the form $\omega \in \Omega^{n-1}(\mathbb{N}\{0\})$

$$
\omega = \sum_{i=1}^{n} (-1)^{i+1} \frac{x^i}{r^n} dx^{N\{i\}},
$$

where

$$r = \sqrt{\sum_i (x^i)^2},$$

and $N = \{1, \ldots, n\}$. Show that $d\omega = 0$ Show using Stokes’ theorem that $H^{n-1}(\mathbb{N}\{0\})$ is non-zero.

**Hint.** First recall that if $\omega = d\xi$ then $i^*(\omega) = i^*(d\xi) = di^*(d\xi)$, where $i : S^{n-1} \rightarrow \mathbb{N}\{0\}$ is the inclusion map. In words: if the form is exact then its restriction is exact. Now let

$$\tilde{\omega} = \sum_{i} (-1)^{i+1} x^i dx^{N\{i\}}.$$

Then $i^*(\omega) = i^*(\tilde{\omega})$ because $r \circ i \equiv 1$. Apply Stokes’ theorem to $\tilde{\omega}$.

**Exercice 15.4.** Recall Green’s theorem

$$
\iint_{\Omega} \left( \frac{\partial F^1}{\partial x^2} - \frac{\partial F^2}{\partial x^1} \right) dx^1 dx^2 = \oint_{\partial \Omega} F \cdot dl
$$

Gauss’ theorem

$$
\iiint_{\Omega} \nabla \cdot F dx^1 dx^2 dx^3 = \iint_{\partial \Omega} F \cdot \nu dS
$$

and Stoke’s theorem (from vector calculus)

$$
\iint_{\Sigma} \nabla \times F \cdot \nu dS = \oint_{\Sigma} F \cdot dl.
$$

See *Analyse avancée pour ingénieurs* Chap. 4,6,7. Identify how each of these is Stokes’ theorem for manifolds by identifying $F$ with a differential form, what are the manifolds of integration and what are the boundaries. Calculate $dF$ in each case.