1. On integrals involving the delta-function

1. Consider the integral

\[ I = \int_{-\infty}^{\infty} dx \, f(x) \delta(ax^2 + bx + c). \]  

Denote the argument of the delta-function by \( g(x) \). There are several possibilities:

- If the equation \( g(x) = 0 \) has no real roots, then the argument of the delta-function is never zero, hence \( I = 0 \).

- Suppose that the equation \( g(x) = 0 \) has two different real roots \( x_{1,2} \). Near each of them the function \( g(x) \) can be written as \( g(x) = g'(x_{1,2})(x - x_{1,2}) + \mathcal{O}((x - x_{1,2})^2) \). Let \( O_1 \) and \( O_2 \) be small neighborhoods of the points \( x_1 \) and \( x_2 \) correspondingly. The integral \( I \) becomes

\[ I = \int_{O_1} \frac{dy}{|g'(x_1 + y)|} f(x_1 + y) \delta(y) + \int_{O_2} \frac{dy}{|g'(x_2 + y)|} f(x_2 + y) \delta(y), \]  

where we made the change of variable \( y = x - x_1 \) in the first integral, \( y = x - x_2 \) in the second integral, and used the property of the delta-function

\[ \delta(\alpha x) = \frac{1}{|\alpha| \delta(x)}, \]  

with \( \alpha \) some constant. Taking the integrals, we have

\[ I = \frac{f(x_1)}{|g'(x_1)|} + \frac{f(x_2)}{|g'(x_2)|}. \]  

Finally, \( |g'(x_1)| = |g'(x_2)| = |a(x_1 - x_2)| = \sqrt{b^2 - 4ac} \), and

\[ I = (f(x_1) + f(x_2))(b^2 - 4ac)^{-1/2}. \]  

- Suppose now that \( x_1 = x_2 = x_0 \). Expanding \( g(x) \) around \( x_0 \) and changing the variable \( y = x - x_0 \), we arrive at

\[ I = \int_{-\infty}^{\infty} dy \, f(x_0 + y) \delta(y) = \lim_{x \to x_0} \frac{f(x)}{2|a(x - x_0)|} = \begin{cases} \infty, & f(x_0) \neq 0, \\ \frac{f'(x_0)}{2|a|}, & f(x_0) = 0. \end{cases} \]
2. Recall that
\[ \delta(f(x)) = \sum_i \delta(x - x_i) \left| f'(x_i) \right|, \tag{7} \]
where \( i \) numerates the roots of the function \( f \). In our case \( E_p = \frac{p^2}{2m} \), and
\[
\int d^3p \, \delta(E_p' - E_p) f(p) = \int d\Omega dp^2 \delta(E_p - E_p') f(p)
= \int d\Omega dp^2 \delta(p - p') \frac{2m}{2p'} f(p)
= mp' \int d\Omega f(n),
\tag{8}
\]
where
\[ n = \frac{|p'| p}{|p|} \tag{9} \]
is a vector of modulus \( |p'| \) in the direction of \( p \).

2. Free particle’s Green function in three dimensions

1. By definition,
\[
\hat{G}_0(z) = \frac{1}{z - H_0}. \tag{10}
\]
This means that
\[
\hat{G}_0(z)|p\rangle = \frac{1}{z - E_p} |p\rangle, \quad E_p = \frac{p^2}{2m}. \tag{11}
\]
Therefore,
\[
\langle x|\hat{G}_0(z)|x'\rangle = \int d^3p \langle x|\hat{G}_0(z)|p\rangle \langle p|x'\rangle = \frac{1}{(2\pi)^3} \int d^3p \frac{e^{ip(x-x')}}{z - E_p}. \tag{12}
\]
2. Let us first compute the radial part of the integral :
\[
\langle x|\hat{G}_0(z)|x'\rangle = 2\pi \frac{1}{(2\pi)^3} \int_0^\infty dp p^2 \int_0^\pi d\theta \sin \theta \frac{e^{ip|x-x'|\cos \theta}}{z - E_p}
= -\frac{1}{(2\pi)^3} \int_0^\infty dp \frac{p^2}{ip|x-x'|} \frac{1}{z - E_p} \left(e^{ip|x-x'|} - e^{-ip|x-x'|}\right)
= \frac{im}{2\pi^2|x-x'|} \int_{-\infty}^\infty dp \frac{e^{ip|x-x'|}}{2mz - p^2}. \tag{13}
\]
The resulting integral can be computed by the method of residues. To this end, we close the contour of integration in the plane of complex \( p \) as shown in figure 1. This does not change the value of the integral, since in the upper half-plane the integrand approaches zero exponentially fast when the radius of the semi-circle goes to infinity. The integrand has two poles at \( p = \pm \sqrt{2mz} \). Recall that \( z = E + i\epsilon, \epsilon > 0 \), hence the pole contributing to the integral is the one at \( p = +\sqrt{2mz} \). Thus,
\[ \langle x|\hat{G}_0(z)|x'\rangle = \frac{im}{2\pi^2|x - x'|^2} 2\pi i \text{ res}_{p = \sqrt{2mz}} \frac{pe^{ip|x-x'|}}{(p - \sqrt{2mz})(p + \sqrt{2mz})} \]

\[ = \frac{m}{2\pi} e^{i\sqrt{2mz}|x-x'|}. \]

3. The formula (12) tells us that the Fourier transform of the function \( G_0(z, x, x') = \langle x|\hat{G}_0(z)|x'\rangle \) is

\[ G_0(z, p, p') = \frac{1}{z - E_p}. \]

This implies in particular the conservation of the free particle momentum. We now use the momentum representation of the Green function to yield

\[ \langle x|(z - \hat{H}_0)\hat{G}_0(z)|x'\rangle = \int d^3p d^3p' d^3p'' \langle x|p\rangle \langle p|z - \hat{H}_0|p''\rangle \langle p''|\hat{G}_0(z)|p'\rangle \langle p'|x'\rangle \]

\[ = \frac{1}{(2\pi)^3} \int d^3p d^3p' d^3p'' e^{ip}\delta(p - p'')(z - E_{p''})\delta(p'' - p') \frac{1}{z - E_{p''}} e^{-ip'\cdot x'} \]

\[ = \frac{1}{(2\pi)^{3/2}} \int d^3p e^{ip(x - x')} = \delta(x - x'). \]

![Fig. 1 – The contour of integration](image)

4. The matrix element \( G_0(z, x, x') \) as a function of the complex variable \( z \) has a branch cut along the real positive values of \( z \). To calculate the difference between the points on the opposite sides of the branch cut, one should continue analytically the function \( \sqrt{z} \) from the one side to another. This gives,

\[ G_0(E + i\epsilon, x, x') - G_0(E - i\epsilon, x, x') = -\frac{m}{2\pi|x - x'|} \left( e^{i\sqrt{2mE}|x-x'|} - e^{-i\sqrt{2mE}|x-x'|} \right) \]

\[ = -\frac{im}{\pi|x - x'|} \sin \left( \sqrt{2mE} |x - x'| \right). \]
5. For large values of $x = |x|$, the modulus $|x - x'|$ can be expanded as

$$|x - x'| = x \sqrt{1 - \frac{2x \cdot x'}{x^2} + \frac{x^2}{x^2}} \approx x - \frac{x \cdot x'}{x}. \quad (18)$$

Thus,

$$G_0(z, x, x') \approx -\frac{m}{2\pi} \frac{\exp \left[ i \sqrt{2mz} \left( x - \frac{x \cdot x'}{x} \right) \right]}{x}, \quad x \to \infty. \quad (19)$$

3. Friedel sum rule

1. The eigenstates of the Hamiltonian $\hat{H}$ form an orthonormal basis of states. This fact allows us to write

$$\hat{G}(x + i\epsilon) = \frac{1}{x - \hat{H} + i\epsilon} = \sum_n \frac{|n\rangle\langle n|}{x - E_n + i\epsilon}. \quad (20)$$

Here by $|n\rangle$ we understand both the bound states and the scattering states. For the matrix element we have,

$$G_{nm}(x + i\epsilon) = \langle n|\hat{G}(x + i\epsilon)|m\rangle = \sum_n \frac{\delta_{nm}}{x - E_n + i\epsilon}, \quad (21)$$

or

$$G_{nn}(x + i\epsilon) = \frac{1}{x - E_n + i\epsilon} = \frac{d}{dx} \log G_{nn}^{-1}(x + i\epsilon) = -\frac{d}{dx} \log G_{nn}(x + i\epsilon). \quad (22)$$

Now we turn to the function $N(x)$. Using the relation

$$\frac{1}{x + i\epsilon} = -i\pi \delta(x) + \mathcal{P} \frac{1}{x}, \quad (23)$$

we have,

$$N(x) = \sum_n \delta(x - E_n) = -\frac{1}{\pi} \sum_n \text{Im} \frac{1}{x - E_n + i\epsilon} = -\frac{1}{\pi} \text{Im} \sum_n G_{nn}(x + i\epsilon). \quad (24)$$

Substitution of the expression (22) then leads to

$$N(x) = \frac{1}{\pi} \text{Im} \sum_n \frac{d}{dx} \log G_{nn}(x + i\epsilon) = \frac{1}{\pi} \frac{d}{dx} \text{Im} \log \det \hat{G}(x + i\epsilon). \quad (25)$$

2. From eq. (25) it follows that

$$N(x) - N_0(x) = \frac{1}{\pi} \frac{d}{dx} [\log \det \hat{G}(x + i\epsilon) - \log \det \hat{G}_0(x + i\epsilon)]$$

$$= \frac{1}{\pi} \frac{d}{dx} \text{Im} \log \det \hat{G}_0^{-1}(x + i\epsilon) \quad (26)$$

Writing $\hat{G}_0^{-1}$ as $|\det \hat{G}_0^{-1}|e^{i\arg \det \hat{G}_0^{-1}}$ gives

$$N(x) - N_0(x) = \frac{1}{\pi} \frac{d}{dx} \arg \det \hat{G}_0^{-1}(x + i\epsilon). \quad (27)$$
4. Slow scattering in a gas

The problem concerns atom-atom scattering inside a gas. The condition

\[ p \lesssim \frac{\hbar}{R} \tag{28} \]

implies \( \mu v_r R \lesssim \hbar \), where \( \mu = \frac{1}{2} m_p \) is the reduced mass of the two atoms, \( v_r = v_1 - v_2 \) is the relative velocity between the two atoms of velocities \( v_1, v_2 \), and \( R = 4 \text{ Å} \). In thermal equilibrium

\[ \frac{1}{2} m_p \langle v^2 \rangle = \frac{3}{2} k T \tag{29} \]

with \( k \) the Boltzmann constant and \( T \) the temperature. The mean-square value of the relative speed \( v_r \) is

\[ \langle v_r^2 \rangle = \langle (v_1 - v_2)^2 \rangle = \langle v_1^2 + v_2^2 - 2(v_1 \cdot v_2) \rangle = 2 \langle v^2 \rangle = \frac{6kT}{m_p} \tag{30} \]

Thus,

\[ \mu R v_r \approx \frac{m_p R}{2} \sqrt{\frac{6kT}{m_p}} \lesssim \hbar \tag{31} \]

i.e.,

\[ T \lesssim \frac{2\hbar^2}{3 m_p c^2} \left( \frac{c}{R} \right)^2 \frac{1}{k} = 2 K \tag{32} \]