1. On validity of the leading-order (LO) WKB approximation

Consider the semiclassical expansion of the wave function
\[ \psi = e^{\frac{i}{\hbar}S}, \quad S = S_0 + \frac{\hbar}{i}S_1 + \left( \frac{\hbar}{i} \right)^2 S_2 + \ldots . \] \hspace{1cm} (1)

It is shown in the lecture notes that one necessary applicability condition of the LO semiclassical approximation
\[ \psi \approx e^{\frac{i}{\hbar}(S_0 + \frac{\hbar}{i}S_1)} \] \hspace{1cm} (2)

is formulated as \(|\lambda'| \ll 1\), where \(\lambda = \hbar/p\) is the de Broglie wave length. However, this requirement is by no means sufficient to ensure the validness of the LO approximation, and in this exercise we have a look at other necessary conditions.

1. Expanding the Schroedinger equation for \(\psi\) up to the second order in \(\hbar\), find \(S'_2\) in terms of \(S'_0\) and \(S'_1\). Rewrite it through the momentum \(p\) and its derivatives; through the energy of the particle \(E\), the potential \(V\) and its derivatives.

2. Show that for the approximation (2) to hold, one must require \(|\hbar S_2| \ll 1\) for all \(n \geq 2\).

3. Show that the first condition in this chain, \(|\hbar S_2| \ll 1\), follows from \(|\lambda'| \ll 1\) and \(\int |\lambda'^2 S_2| \, dx \ll 1\).

4. From the condition \(|\lambda'| \ll 1\) obtain the following inequality,
\[ \left| \frac{\delta A}{T_{kin}} \right| \ll 1 , \] \hspace{1cm} (3)

where \(\delta A\) is a work done by a force \(F = -V'\) on a distance \(\lambda\), and \(T_{kin}\) is a kinetic energy of the particle.

2. On accuracy of the LO WKB approximation

Consider the particle of unit mass and with the energy \(V_0\), moving in the potential
\[ V(x) = \begin{cases} 0, & x < 0, \\ V_0 \sqrt{\frac{x}{x_0}}, & x > 0. \end{cases} \] \hspace{1cm} (4)

1. Find the LO WKB wave function of the particle in the region \(x > x_0\).
2. Find how small (or large) one should take \(V_0\) to be sure that the LO approximation of the wave function is accurate to 1 percent for all \(x > 2x_0\).
3. On asymptotics of the potential in the WKB approximation

Consider the particle of zero energy, moving in the potential $V(x)$ shown schematically on figure 1. Assume that $V(x)$ approaches a constant negative value at $x \to -\infty$, and that its behavior at large positive $x$ is of the form

$$V(x) \sim x^{-n}, \quad n > 0, \quad x \to \infty.$$  \hspace{1cm} (5)

1. Give a constraint on possible values of $n$, that ensures the validness of the LO WKB approximation of the decaying wave function in the limit $x \to \infty$.

Suppose now that the asymptotics of the potential at large positive $x$ is

$$V(x) \sim \left(\frac{\log x}{x}\right)^2.$$  \hspace{1cm} (6)

2. Is it legitimate to use WKB theory to predict the large-$x$ behavior of the decaying wave function?

4*. WKB expansion beyond the LO

The semiclassical approximation of the amplitude (2) can be continued beyond the LO by successive calculation of higher-order terms in the expansion (1). These terms exhibit some interesting properties which, as we will see later, are important when computing the energy levels of the particle.

1. Obtain an iterative expression for $S'_n$ through $S'_0, ..., S'_{n-1}$.

2. Show that all odd terms $S'_{2k+1}$ are
   (a) real (and, hence, do not contribute to the phase of the wave function),
   (b) total derivatives.
5*. One approach to matching the WKB wave functions

Consider the general normalized LO WKB solution of the Schroedinger equation in the classically allowed \((x < a)\) and classically forbidden \((x > a)\) regions:

\[
\psi(x)_{x < a} = C_1 \frac{1}{\sqrt{p}} \exp \left( \frac{i}{\hbar} \int_x^a p \, dx \right) + C_2 \frac{1}{\sqrt{p}} \exp \left( -\frac{i}{\hbar} \int_x^a p \, dx \right), \quad x < a,
\]

\[
\psi(x)_{x > a} = C \frac{1}{\sqrt{|p|}} \exp \left( -\frac{1}{\hbar} \sqrt{|p|} \int_x^a p \, dx \right), \quad x > a.
\]

(7)

To build a single smooth solution out of (7), one should establish proper relations between \(C\) and \(C_1, C_2\). To this end, one can find an exact solution to an approximate Schroedinger equation in the small region containing the turning point, where the potential can be approximated by a linear function,

\[
V(x) \approx V(a) + V'(a)(x - a).
\]

(8)

This solution is then matched with the functions (7) at the points where the approximations (2) and (8) are both valid. Given below are the exercises suggested to accomplish this program.

1. Find the general solution \(\tilde{\psi}(x)\) of the Schroedinger equation with the linear potential (8).
   *Hint*: It can be expressed through Airy functions.

2. Find the regions of \(x\) where both the LO WKB functions (7) and the solution found above are valid approximations to the exact solution.

3. Comparing asymptotic behavior of \(\tilde{\psi}(x)\) in those regions with the functions (7), find the relation between \(C\) and \(C_1, C_2\).