QUESTION SET 8

Exercise 1: Continuous population model

We study a system with lateral connection \( w(x - y) \) given by:

\[
\tau \frac{\partial h(x,t)}{\partial t} = -h(x,t) + \int w(x - y) F[h(y,t)] dy + I_{ext}(x,t),
\]

where \( F[h(x,t)] = A(x,t) \) is the population’s activity at the point \( x \) at time \( t \).

1.1 Show that, for a constant current \( I_{ext} \), there exists a homogeneous stationary solution \( h(x,t) = h_0 \) with a constant activity \( A_0 \) given by:

\[
A_0 = F(h_0) = h_0 - \frac{I_{ext}}{\bar{w}},
\]

with \( \bar{w} = \int w(x - y) dy \).

1.2 We set \( h(x,t) = h_0 + \Delta h(x,t) \) where \( \Delta h \) is a small perturbation. Linearize equation (1) around \( h_0 \) by substituting \( h(x,t) = h_0 + \Delta h(x,t) \) and Taylor-expanding \( F[h(x,t)] \) until first order. Apply a spatial Fourier transform (with respect to \( x \)) and use the convolution theorem to simplify the resulting temporal differential equation. Solve this differential equation and perform the inverse Fourier transform on the solution to obtain \( \Delta h(x,t) = \int g(k) dk \) where

\[
g(k) = C(k)e^{ikx}e^{-\kappa(k)t/\tau}.
\]

Identify the function \( \kappa(k) \). For which values of \( k \) do we get \( \kappa < 0 \)? What does this mean for the stability of the solution \( \Delta h(x,t) \)?

1.3 Consider:

\[
w(z) = \sigma_2 e^{-z^2/(2\sigma_1^2)} - \sigma_1 e^{-z^2/(2\sigma_2^2)} \]

\[
\sigma_1 = 1 \text{ and } \sigma_2 = 10.
\]

Sketch the qualitative behaviour of \( w(z) \) and

\[
\int w(z) \cos(kz) dz.
\]

Determine graphically the stability condition.

Exercise 2: Stationary state in a network with lateral connections

Consider a neural network with lateral connections represented in figure 1: the interaction is locally excitatory and long range inhibitory:

\[
w(x,x') = \begin{cases} 
1 & |x - x'| \leq \sigma \\
-b & |x - x'| > \sigma
\end{cases}
\]

Therefore \( \sigma \) corresponds to the range of the excitatory connections. The activity \( A \) of a neuron at position \( x \) is given by:

\[
A(x) = F[h(x)],
\]
where $h(x)$ is the total potential of the neuron at position $x$, defined as:

$$ h(x) = \int w(x, x') A(x') dx' + I_{ext}(x). $$  \hspace{1cm} (4)

The function $F(h)$ is a simple threshold function:

$$ F(h) = \begin{cases} 
1 & h > \Theta \\
0 & h \leq \Theta 
\end{cases} $$  \hspace{1cm} (5)

In this exercise we do not add any external input i.e. $I_{ext}(x) = 0$. The aim of the exercise is to find the neural activity $A(x)$. In order to do so, we assume that $A(x)$ may have a rectangular shape (of breadth $2d$ and amplitude 1, as shown in figure 1) and we prove this assumption with the following passages.

2.1 Consider a point at location $x_0$ close to $x = 2d$ and calculate its input potential, assuming that $2d > \sigma$.

(Hint: there is excitatory input from the right and there is excitatory and inhibitory input from the left).

2.2 Exploit that at $x_0 = 2d$ we must have $h(x_0) = \Theta$. Why? Calculate $d$.

2.3 Convince yourselves that the bump of size $2d$ is therefore a solution for the activity $A(x)$ and discuss its properties.

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**Exercise 3: Stability of the stationary bump solution**

We keep the same type of network as before and assume that the network rests in the stationary bump solution $A(x)$ determined in the previous exercise. We now investigate the stability of that solution. In order to do so, we perturb the system at time $t_p$ by slightly changing the width $D(t)$ of the activity bump $A(x, t_p)$ from $2d$ to $2d + \delta$, with some $\delta = \delta(t_p) \ll d$ as indicated in fig. 2. Use the following steps to study the stability of the unperturbed solution:

3.1 Discretize time in the underlying differential equation,

$$ \frac{\partial h(x, t)}{\partial t} = -h(x, t) + \int w(x - y) F[h(y, t)] dy, $$  \hspace{1cm} (6)
using $\Delta t = \tau$. Solve for $h(x, t + \Delta t)$ on the left hand side.

3.2 Using the discretized equation, calculate the potential $h(x, t_p + \Delta t)$ one time step after the perturbation by explicitly evaluating the integral with the perturbed (broadened) activation $A(x, t_p)$. Consider again a position $x_0$ close to $x = 2d$

3.3 Evaluate the potential at $x_0 = 2d + \delta$. Is it below or above the threshold $\Theta$? What does this mean for the evolution of the bump width $D(t)$ in the next time step(s)? Based on this, discuss the stability of the stationary solution $A(x)$.

3.4 Derive an iteration formula for the perturbation length $\delta(t_p + \Delta t)$ after one time step. Justify (e.g. via complete induction) that the derived formula holds for all following time steps and derive an explicit formula for $\delta(t_p + n \cdot \Delta t)$, where $n$ is the number of time steps. What is the asymptotic value of $\delta$ for $n \to \infty$ and what is the functional form of the underlying decay?

![Diagram](image-url)

Figure 2: Illustration of the slight perturbation of the activity bump.