Neural Networks and Biological Modeling
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Questions set 4

Exercise 1: Inhibitory rebound

Consider the following two-dimensional Fitzhugh-Nagumo model:

\[
\begin{align*}
\frac{du}{dt} &= u(1-u^2) - w + I \equiv F(u,w) \\
\frac{dw}{dt} &= \varepsilon (u - 0.5w + 1) \equiv \varepsilon G(u,w),
\end{align*}
\]

(1)

where \(\varepsilon \ll 1\).

1.1 Suppose that an inhibitory current step is applied, \(I(t) = \begin{cases} -I_0 & t \leq 0 \\ 0 & t > 0 \end{cases}\)

How does the fixed point move?

1.2 What happens after the driving current is removed? Sketch the form of the trajectories for increasing values of \(I_0\). What happens for large \(I_0\)?

Exercise 2: Phase Plane Analysis

In this exercise, we use the phase plane to study the dynamics of a two dimensional, nonlinear neuron model. The system is described by:

\[
\begin{align*}
\frac{du}{dt} &= F(u,w) \\
\frac{dw}{dt} &= G(u,w)
\end{align*}
\]

(2)

where \(F(u,w) = f(u) - w + I(t)\) and \(G(u,w) = \varepsilon (g(u) - w)\) with \(\varepsilon = 0.1\). \(I(t)\) is an external current.

Figure 1 shows the \(u\)- and \(w\)-nullclines for the case \(I(t) = 0\):

2.1 Given \(F(u_4,0) = 5, G(u_4,0) = 1\), draw a few flow arrows along the two nullclines in figure 1.

2.2 Without doing any computation, can you determine the stability of the fixed point 2 (the one at \((u_2, w_2)\))? Justify your answer.

2.3 Discuss the stability of the third fixed point (the one at \((u_3, w_3)\)) analytically. That is, linearize the system at the fixed point 3 and discuss the evolution of a small perturbation around that point. For the numeric calculations, use \(\varepsilon = 0.1\) and approximate the values of \(\frac{df}{du}|_{u_3}\) and \(\frac{dg}{du}|_{u_3}\) from figure 1.

2.4 Assume the neuron is at rest. Then, at \(t_0\) we apply a pulse stimulus \(I(t)\) to this system:

\[I(t) = (u_3 - u_1)\delta(t - t_0)\]
Figure 1: Phase Plane, $I(t) = 0$

(i) Sketch the trajectory $(u(t), w(t))$ in Figure 1.

(ii) Sketch the membrane potential $u(t)$ vs. time in a new figure. Make sure you get the two plots qualitatively correct: Clearly indicate important states, for example at $t < t_0$, at $t_0$, and at $t > t_0$. Furthermore, in your $u(t)$ plot, fast and slow regions should be distinguishable.

2.5 Refering to figure 1, discuss the effect of injecting pulse currents $I(t) = q\delta(t - t_0)$ of different amplitudes $q$ into the neuron. What happens if we gradually increase $q$? Does this neuron model have a threshold?

2.6 Assume the neuron is at rest. We then apply a step current to the neuron:

$$ I(t) = \begin{cases} 
0 & \text{if } t \leq 0 \\
3 & \text{if } t > 0 
\end{cases} $$

(i) Sketch the nullcline $\frac{du}{dt} = 0$ for $t > 0$ in figure 2.

(ii) In figure 2, mark the state of the system at $t = 0$. Starting from that state, sketch the trajectory of the system for $t > 0$.

(iii) Qualitatively discuss the evolution of the system for $t \to \infty$. 
Exercise 3: Impulse response

Consider the following system with separation of time scales:

\[
\begin{align*}
\frac{du}{dt} &= f(u) - w + I \\
\frac{dw}{dt} &= \varepsilon (bu - \gamma w)
\end{align*}
\]

where \(\varepsilon \ll 1\) and

\[f(u) = \begin{cases} 
-u & \text{if } u < 1 \\
\frac{a-1}{a} - 1 & \text{if } 1 \leq u < 1 + 2a \\
2(1+a) - u & \text{if } u > 1 + 2a
\end{cases}\]

Assume that \(b, \gamma, a > 0\) and \(b/\gamma > 1/a\). Discuss the behaviour of the trajectories of \(u(t)\) in response to a current pulse \(I(t) = q\delta(t)\). Sketch these trajectories in the phase plane and in the temporal domain for a few values of \(q\). Does the model exhibit a threshold-like behaviour?