Measuring systems

Lecturer: Andras Kis

In class demo: RTD and photodetector

USB connector

Arduino UNO board

Conditioning circuit

$U_0 = 5 \text{ V}$

$U_{\text{out}}$

$R_{\text{load}} = 1k$

PT 100 Sensor

$U_0$ $R_{\text{load}}$ $U_{\text{out}}$

$0V$ $R_{\text{sensor}}$

$U_{\text{out}}$
In class demo: RTD and photodetector

Data analysis (recording, averaging, etc.)

1.3 Arduino UNO board Conditioning circuit

Sensor

Rload

U0

0V

Rsensor

Uout

U0 = 5 V

Noise reduction

Acquisition (Analog – digital conversion)

Arduino UNO board

Conditioning circuit

Measurement chain

Chapter 1

Sensor ➔ Conditioning

Chapter 2

Modeling

Chapter 3

Noise reduction and signal processing ➔ Acquisition

Chapter 4

Action

Data analysis Comparison

Chapter 5 and 6
Measuring systems

• Sensors and their conditioning
• Modeling sensors
• Noise estimation and reduction
• Data acquisition
• Data analysis and treatment
• Comparison between measurement results

References

• Georges Asch, Acquisition de données, Dunod, 2003
• Ph. Robert, TE vol 17, Systèmes de mesure

• Transparencies
• Exercises + solutions
Organisation

• Room BC 01
• Exercises
  - 12 problem sets
  - Discussions during the exercises
  - Work at home
• Written mock exam (end November or early December) – bonus (max +1 on the final exam)
• Written exam
• Prerequisites: Electrotechnique 1 and 2
• Needed for: TP Measuring systems

Expected work load

• 1 credit = 30 work hours (source: EPFL, CRAFT)
• 3 credits x 30 = 90 hours total
  - 10 h preparation for the exam
  =80h
  -3x14 lectures + exercises
  =38 h for individual work at home
  =2.5-3 h/week
Chapter 1: Sensors and conditioning circuits

**Sensors and conditioning circuits**

- **Introduction**
  - Transducer: sensor, actuator

- **Passive sensors and their conditioning**
  - Temperature – RTD (resistance temperature detector)
  - Displacement – capacitive sensors
  - Displacement – inductive sensors
  - Light intensity – photoconductors

- **Active sensors and their conditioning**
  - Temperature – thermocouple (thermoelectric effect)
  - Light intensity – photovoltaic cell (photovoltaic effect)
  - Displacement – piezoelectric gauge (piezoelectric effect)
A transducer is an element that converts one physical quantity into another physical quantity:
- Mercury thermometer (temperature – displacement)
- Accelerometer (acceleration – voltage)
- Electrode in a battery (ion – electrical charge)
- Motor (electrical current – mechanical moment)
- LED (electrical current – light)

Sensor - actuator

• A sensor is a transducer that converts a physical quantity into an electrical quantity:
  - Resistance thermometer (temperature - resistance)
  - Photodetector (light - current)

• An actuator is a transducer that converts an electrical quantity into a non-electrical quantity
  - Piezo actuator (charge - displacement)
  - Resistive heater (current - heat)
  - LED (current - light)
Sensors

- Sensitivity $S$: response in magnitude
- Transfer function: frequency response
- Noise: sensitivity to perturbations (internal and external)
### Passive and active sensors

- **Passive sensors** - require an external power source
  - Examples:
    - Resistive thermometer
    - Capacitive displacement sensor

- **Active sensors** - generate the electrical signal from the measured quantity
  - Examples:
    - Thermocouples – thermoelectric effect
    - Accelerometers – piezoelectric effect
Passive sensors

<table>
<thead>
<tr>
<th>Measured quantity</th>
<th>Sensitive characteristic</th>
<th>Device</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature</td>
<td>Resistance</td>
<td>RTD (resistance temperature detector)</td>
</tr>
<tr>
<td>Mechanical (force, pressure, acceleration, vibrations, sound level, displacement)</td>
<td>Resistance, capacitance, inductance</td>
<td>potentiometer, microphone LVDT (linear variable differential transformer), accelerometer, strain gauge</td>
</tr>
<tr>
<td>Light intensity</td>
<td>Resistance</td>
<td>photoconductor phototransistor</td>
</tr>
</tbody>
</table>

Resistive temperature sensors (RTD)

- Resistance of a metal as a function of temperature:
  \[ R = R_0 \cdot f \left( T - T_0 \right) \]
  \[ R \quad \text{– Resistance at temperature } T \]
  \[ R_0 \quad \text{– Resistance at temperature } T_0 \]

- For platinum (PT100):
  \[ R(T) = R_0 \left( 1 + A(T - T_0) + B(T - T_0)^2 \right) \]
  \[ T \quad \text{– temperature in °C} \]
  \[ T_0 = 0 \text{°C} \]
  \[ R_0 = 100 \text{Ω} \]
  \[ A = 3.9 \times 10^{-3} \text{ °C}^{-1} \]
  \[ B = -5.775 \times 10^{-7} \text{ °C}^{-2} \]

- Linear but low sensitivity
Resistive temperature sensors (Thermistors)

- Ceramics or polymers
- Generally described by Steinhart-Hart equation:
  \[ \frac{1}{T} = A + B \ln(R) + C (\ln(R))^3 \]
  
  \[ T \quad \text{– Temperature in Kelvin} \]
- Example: Omega 44006
  
  \[ R_{T=25^\circ C} = 10000 \, \Omega \]
  \[ A = 1.032 \times 10^{-3} \, ^\circ C^{-1} \]
  \[ B = 2.208 \times 10^{-4} \, ^\circ C^{-1} \]
  \[ C = 1.276 \times 10^{-7} \, ^\circ C^{-1} \]
- Non-linear but high sensitivity

Semiconducting diode thermometers

- Si, Ge, etc.
  - pn junctions
  - inexpensive and (mostly) linear
  - limited temperature range
    (-50 – 150 °C)
Conditioning circuits for resistive sensors

**Voltage divider**

\[ U_{out} = U_0 \frac{R_{\text{sensor}}}{R_{\text{load}} + R_{\text{sensor}}} \]

For \( R_{\text{load}} = R_{\text{sensor}} = R \):

\[ U_{out} = \frac{U_0}{2} \]

---

**Wheatstone bridge**

\[ U_{out} = \left[ \frac{R_{\text{sensor}}}{R_{\text{sensor}} + R} - \frac{R}{2R} \right] U_0 \]

For \( R_{\text{sensor}} = R \):

\[ U_{out} = 0 \]
Displacement sensor - resistive

- Potentiometer
  - Resistor with a sliding contact
  - Acts as a voltage divider

\[ U_{out} = \frac{R_x}{R} U_0 \]

Displacement sensor - inductive

- LVDT (Linear Variable Differential Transformer)

\[ V_o = k \cdot V_{in} \cdot x \]
Mutual inductance – differential transformer

\[ u_1 = R_i i_1 + L_i \frac{di_1}{dt} + (M''-M') \frac{di_2}{dt} \]

\[ u_2 = -(R'_2 + R''_2) i_2 - (L'_2 + L''_2) \frac{di_2}{dt} + (M''-M') \frac{di_1}{dt} \]

\[ U_1 = (R_i + j \omega L_i) L_1 + j \omega (M''-M') L_2 \]

\[ U_2 = -(R'_2 + R''_2 + j \omega L'_2 + j \omega L''_2) L_2 + j \omega (M''-M') L_1 \]

For \( R_c \gg \), \( i_2 \approx 0 \)

\[ U_2 = \frac{j \omega [M''(x) - M'(x)]}{R_i + j \omega L_i} U_1 \]

\[ M'(x) = M(0) + ax + bx^2 + ... \text{ for } x > 0 \]

\[ M''(x) = M(0) - ax + bx^2 + ... \text{ for } x < 0 \]

2\textsuperscript{nd} order approximation:

\[ M''(x) - M'(x) = -2ax \]

We get a linear relationship:

\[ U_2 = \frac{-2 j \omega \cdot a U_1}{R_i + j \omega L_i} X \]

Voltage on \( L_2' \) due to current \( i_2 \):

\[ L'_2 \frac{di_2}{dt} \]

Voltage on \( L_2' \) due to current \( i_1 \):

\[ M' \frac{di_1}{dt} \]
Capacitive displacement sensor

- Capacitance

\[ C = \varepsilon \frac{A}{d} \]

- Microphone: sound (external pressure variations) cause the membrane to vibrate (displacement \( dx \))

\[ i = -\frac{dq}{dt} \]

\[ U = -R \frac{dq}{dt} \]

Conditioning for capacitive sensors

Pressure sensor

flexible metallic membrane
Strain gauge

- Principle: change in resistance upon mechanical deformation

\[ R_{\text{initial}} = \rho \frac{l}{S} \]

\[ R_{\text{strained}} = (\rho + \Delta \rho) \frac{l + \Delta l}{S + \Delta S} \]

- \( R \): resistance
- \( \rho \): resistivity
- \( l \): length
- \( S \): cross-sectional area
Strain gauge

\[ \frac{\Delta R}{R} = K \frac{\Delta l}{l} \]

Gauge factor: \( K \approx 2 - 4 \)

Strain:

\[ \varepsilon = \frac{\Delta l}{l} \]

\[ U_{out} = f(U_0, \Delta R/R) \]

- **Let strain** \( \varepsilon \) **be the relative change in length and stress** \( \sigma \) **the force** \( F \) **per cross-sectional area** \( S \):

\[ \varepsilon = \frac{\Delta l}{l} \quad \sigma = \frac{F}{S} \]

- **Strain and stress are related through the Young’s modulus** \( Y \) **and Poisson ratio** \( \nu \):
  - In the direction parallel to the stress:
    \[ \varepsilon_{||} = \frac{\sigma}{Y} \]
  - Perpendicular to the stress:
    \[ \varepsilon_{\perp} = -\nu \varepsilon_{||} = -\nu \frac{\sigma}{Y} \]
Strain gauge

• Surface change:

\[ \frac{\Delta S}{S} = -2\nu \frac{\Delta l}{l} \]

• Resistance change:

\[ \frac{\Delta R}{R} = \frac{\Delta \rho}{\rho} + \frac{\Delta l}{l} - \frac{\Delta S}{S} \]

\[ \frac{\Delta R}{R} = (1 + 2\nu) \frac{\Delta l}{l} + \frac{\Delta \rho}{\rho} \]

**Dominant terms**

Metals: first term (geometry)
Semiconductors: second term
Force sensor

- Based on a strain sensor attached to a test object

\[
F = \frac{\Delta l}{l} = \frac{F}{A \cdot Y}
\]

\[
\Delta R = K \frac{\Delta l}{l}
\]

\[
\frac{\Delta U}{U} = \frac{\Delta R}{R} I
\]

\[
\Delta U = K \frac{F}{Y \cdot A} I
\]

Sensors for force, pressure, acceleration

Force \( F \)

Pressure \( P = P_1 - P_2 \)

Acceleration \( a \)

\[
F = f(\varepsilon)
\]

\[
P_1 > P_2\]

\[
P = f(\varepsilon)
\]

\[
a = \Delta x \cdot \frac{k}{m} = f(\varepsilon)
\]
Some applications for accelerometers

1. Nintendo Wii
2. iPhone etc.
3. Airbag
4. Image stabilisation

MEMS-based accelerometer

- Micro-Electro-Mechanical systems: integration of electronics and mechanical elements: sensors and actuators

ADXL202 accelerometer

www.analog.com
Microelectromechanical systems (MEMS)

- Micro-Electro-Mechanical systems: integration of electronics and mechanical elements: sensors and actuators

Movement of the beam controlled by springs with spring constant $k$

ADXL202 accelerometer
Analog Devices website

Force on a mass $m$ subject to acceleration $a$:

$$ F = ma $$

Restoring force from the spring:

$$ F = k \cdot \Delta x $$

So the deflection is:

$$ \Delta x = \frac{m}{k} a $$

It is read out by measuring the electrical capacitance between the « fingers »

$$ C = C(x) $$
Light intensity measurements

• Photoconductor
  - Highly resistive semiconductor (for example CdS)
  - Under illumination, electron-hole pairs are excited and the resistance decreases
  - Requires a voltage source to operate in a similar way to RTDs

Light intensity measurements

• Phototransistor
  - npn or pnp junction
  - Light absorbed in the base-collector junction generates electrons that are injected into the base and amplified by the transistor’s current gain
  - Higher responsivity (A/W) but longer response time and higher dark currents than photodiodes and photoconductors
Active sensors

Temperature – Thermoelectric effect

• Seebeck effect – temperature difference results in a potential difference

\[ T_A < T_B \rightarrow e^- \text{ in B are more energetic than in A} \]
\[ e^- \text{ move from B to A} \rightarrow \text{more electrons in A} \]
\[ \rightarrow U_{AB} > 0 \]

• Thomson effect – heat transport due to electrical current

\[ e^- \text{ move from B to A} \rightarrow \text{energy loss} \rightarrow \text{temperature increase in the middle of the conductor} \]
\[ e^- \text{ move from A to B} \rightarrow \text{energy is absorbed} \rightarrow \text{temperature decrease in the middle} \]
Temperature – Thermoelectric effect

• Peltier effect

- The energy of an electron depends on the temperature, work function (type of the conductor) and local electromagnetic field
- By passing from 1 to 2, the energy of an electron is modified, resulting in heat being absorbed (cooling) or generated (heating)

• Thermoelectric effect - common name for these three effects
• Sensor: thermocouple
• Actuator: Peltier element

Thermocouple

- sensitivity
- characteristic of the AB metal pair
- typically 10-100 μV/K

Most common – type K:
- chromel (90% Ni, 10% Cr) / alumel (95% Ni, 2% Mn, 2% Al, 1% Si)
- $S = 41\,\mu V/K$ at room temperature

• Practical devices have built-in cold junction compensation
Thermocouples vs. RTD

<table>
<thead>
<tr>
<th>Characteristic</th>
<th>Thermocouples</th>
<th>RTD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Range</td>
<td>Up to 2300°C</td>
<td>Up to 500°C</td>
</tr>
<tr>
<td>Speed</td>
<td>Fast (&lt;1 s)</td>
<td>Slow (&gt;1 s)</td>
</tr>
<tr>
<td>Precision</td>
<td>Low (~1 °C)</td>
<td>High (&lt;&lt;1 °C)</td>
</tr>
</tbody>
</table>

Displacement – Piezoelectric effect

Occurs in materials with no inversion symmetry

\[ q = d\sigma \]

- \( q \) – induced charge
- \( d \) – piezoelectric coefficient
- \( \sigma \) – mechanical stress
Displacement – Piezoelectric effect

• Sources of mechanical stress
  - Force, deformation, vibration, sound

• Materials
  - Quartz, ceramics (PZT), PVDF (Polyvinylidene fluoride)

• Applications
  - Force and pressure sensors
  - Accelerometers
  - Microphones
Light intensity measurements - Photodiode

- Light is absorbed in a pn junction
- Photoexcited charge carriers are separated in the internal electric field
- Voltage is generated
- Non-linear response

Key Points

- There is a large number of sensors and measurement principles
- Passive sensors - based on measurements of $R$, $L$, $C$; require a power supply
- Active sensors – directly use the measured quantity for generating the signal
- The signal is obtained with the use of a conditioning circuit
- When choosing an appropriate sensor, keep in mind the operating principle, the measurement range, possible sources of errors
Chapter 2: Modelling

Measurement chain

- **Data analysis (recording, averaging, etc.)**

- **Acquisition** (Analog – digital conversion)

- **Conditioning**

- **Noise reduction**

- **Modeling**

Arduino UNO board

Conditioning circuit
Chapter 2: Modeling

- Introduction
- General model
- Static transfer
- Dynamic transfer
What are the important parameters of a measurement system?

**SPECIFICATIONS**
- **Excitation:** 10 to 50 N: 5 Vdc
  ≥100 N: 10 Vdc
- **Output:** 10 to 50 N: 1.5 mV/V (nom)
  ≥100 N: 2 mV/V (nom)
- **Accuracy:** (Linearity and
  Hysteresis combined)
  ±0.15% FSO ≤500 N
  ±0.20% FSO ≥1000 N
- **Repeatability:** 0.20% FSO
- **5-Point Calibration:** (in tension)
  0%, 50%, 100%, 50%, 0%
- **Zero Balance:** ±2% FSO
- **Deflection:** 0.025 to 0.075 mm

- **Operating Temp Range:**
  -54 to 121°C (-65 to 250°F)
- **Compensated Temp Range:**
  16 to 71°C (60 to 160°F)
- **Thermal Effects:**
  Span: ±0.009% FSO/°C
  Zero: 0.009% FSO/°C
- **Safe Overload:** 150% of Capacity
- **Ultimate Overload:** 300% of Capacity
- **Bridge Resistance:** 350 Ω minimum
- **Construction:** Stainless Steel
- **Electrical Connection:** 1.5 m (5 ft)
  4 Conductor, Shielded Cable;
  ≤50N: SS Overbraided with temperature
  compensation board

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**Force sensor**

Force sensor: strain gauge on a steel bar

\[ \frac{\Delta R}{R} \bigg|_F = K \varepsilon = \frac{a}{Y} F \]

- \( K \varepsilon \) – proportionality factor
- \( Y \) – Young's modulus
- \( a/Y \) – Sensitivity

**Resistance**

- 120 Ω

**Temperature**

- 120 Ω

**Force**

- 12 N
**Force sensor**

- Influence of the force (measured quantity) on the sensor output:
  \[
  \frac{\Delta R}{R} = K \varepsilon = \frac{a}{Y} F
  \]

- Influence of the temperature
  \[
  \frac{\Delta R}{R}_{T,\text{offset}} = \alpha_T \cdot \Delta T
  \]
  Resistance changes with the temperature:
  **added offset on the left-hand side**
  \[
  \frac{\Delta Y}{Y}_T = -\alpha_Y \Delta T, \quad \alpha_Y = 0.26 \times 10^{-3} \, ^\circ\text{C} \quad \text{(steel)}
  \]
  Material's stiffness (Young’s modulus of the steel bar) changes with the temperature:
  **modified slope**

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\( x_i \) – interfering input
- adds an offset

\( x_m \) – modifying input
- changes the slope (sensitivity)

\( y \) – measured quantity
General model

\[ x_d \rightarrow F_d \rightarrow \Delta m F_d \rightarrow y \]
\[ x_i \rightarrow F_i \rightarrow \Delta m F_i \rightarrow y \]

\[ x_d \rightarrow \text{desired quantity} \]
\[ x_i \rightarrow \text{interfering input} \]

\[ F_d, F_i \rightarrow \text{transfer functions} \]
\[ \Delta m F_d, \Delta m F_i \rightarrow \text{parameters of } F_d, F_i \]

\[ y(x_d, x_i, x_m) \]
\[ y(x_d) \]
\[ y(x_i) \]
Examples

- Temperature and a strain gauge
- Temperature and a photoresistor
- Temperature and humidity and a pressure sensor (changes in the membrane properties)

Example

- Knowing the modifying and interfering inputs can help us compensate their influence
- Example: strain gauge

\[
R_j = R \text{ in the absence of deformation}
\]

\[
\left. \frac{\Delta R_j}{R_j} \right|_F = K \varepsilon
\]

sensitivity

\[
U_{out} = \frac{\Delta R}{4R} U_0 = \frac{1}{4} K U_0 \cdot \varepsilon = S \cdot \varepsilon
\]

Influence of \( T \)

\[
\frac{\Delta R_j}{R_j} = \alpha_R \Delta T \rightarrow U_{out,T} = \alpha_R \Delta T \frac{U_0}{4} + \frac{1}{4} K U_0 \cdot \varepsilon
\]
Example

• Knowing the modifying and interfering inputs can help us compensate their influence

• Example: strain gauge

\[ \text{Influence of } U_0 \]
\[ U_{out} = \frac{\Delta R}{4R} (U_0 + \Delta U_0) \]
\[ \Delta S = \frac{1}{4} K \Delta U_0 \]

Static transfer characteristics

• Acquisition of static transfer characteristics:
  - All the inputs are kept constant, except for one which is varied stepwise (static calibration)
  - Measure the output after all the transients have disappeared
  - This is called static calibration

• Components of the static transfer characteristics:
  - Range
  - Sensitivity
  - Offset
  - Drift
  - Linearity
  - Hysteresis
  - Repeatability
  - Resolution
  - Threshold
  - Stability
**Measurement range**

- **Input range (input span):** the interval within which the input values can vary
  \[ \text{Range} = x_{\text{max}} - x_{\text{min}} \]

- **Output range (output span)**

- **Measurement range** – sometimes also called full scale (FS)

**Sensitivity**

- **Calibration curve**
  - Linear
  - Non-linear

- **Sensitivity**
  \[ S = \frac{dy}{dx} \bigg|_{x=x_0} \]
Offset

- Value of the output \( (y_0) \) for input \( x = 0 \)
  \[ y = S \cdot x + y_0 \]

- Error in terms of the input value:
  \[ x_{offset} = \frac{y_0}{S} \]

Example: pressure sensor

Sensitivity: \( S = (14\text{mA-4mA})/100 \text{ bar} = 0.1 \text{ mA/bar} \)

Offset: \( y_0 = 4\text{mA (40 bar)} \)
Example: specification sheet for a pressure sensor

<table>
<thead>
<tr>
<th>Pressure range</th>
<th>FS</th>
<th>0-200 kPa</th>
</tr>
</thead>
<tbody>
<tr>
<td>Offset</td>
<td>Voff</td>
<td>±2 mV</td>
</tr>
<tr>
<td>Sensitivity</td>
<td>ΔV/ΔP</td>
<td>0.2 mV/kPa</td>
</tr>
</tbody>
</table>

Drift

- Slow (in terms of time) variations of the sensitivity or offset
  - Example: instrument warming up
  - Repeated measurements give successively lower or higher results than previous measurements
  - Can be checked by repeatedly performing zero readings

\[ \Delta y_o(C_2) - \Delta y_o(C_1) \]

- Condition 1: the perturbation changes the offset (interfering)
- Condition 2: the perturbation changes the sensitivity (modifying)
Example of drift

• Influence of the temperature on a pressure sensor

Offset drift: self-heating of the strain gauge (resistor)
Sensitivity drift: heating of the membrane resulting in the change of the Young’s modulus

Linearity

• Describes in what measure the sensitivity independent of the measurand (input value)

\[ y = y_o + S \cdot x \pm (\Delta_1 Sx \text{ or } \Delta_2 Sx_{\text{max}}) \]

whichever is bigger
Reference line – least squares fit

• Assume

\[ y_e = ax + b \]

• Minimize total distance

\[ D = \sum_{i=1}^{N} (y_i - ax_i - b)^2 \]

\[ \frac{\partial D}{\partial a} = 0, \quad \frac{\partial D}{\partial b} = 0 \]

\[ a = \frac{\sum_{i=1}^{N} x_i y_i - \frac{1}{N} \sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i}{\sum_{i=1}^{N} x_i^2 - \frac{1}{N} \left( \sum_{i=1}^{N} x_i \right)^2} \]

\[ b = \bar{y} - a \bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i^2 \sum_{i=1}^{N} y_i - \frac{1}{N} \sum_{i=1}^{N} x_i \sum_{i=1}^{N} x_i y_i}{\sum_{i=1}^{N} x_i^2 - \frac{1}{N} \left( \sum_{i=1}^{N} x_i \right)^2} \]

Example of linearity

**Output (mA)**

**Actual response**

**Idealised response**

\[ \text{Linearity} \% = \frac{\pm \Delta x_{\text{max}}}{\text{FS}} \]
Hysteresis

- Response depends on history
  - Magnetic polarisation
  - Piezoelectric polarisation
  - Friction

\[ Hysteresis(\%) = \frac{\pm \Delta x_{\text{max}}}{FS} \]

Example: pressure sensor datasheet

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<tbody>
<tr>
<td>Offset</td>
<td>( V_{\text{off}} )</td>
<td>( \pm 1 )</td>
<td>mV</td>
</tr>
<tr>
<td>Sensitivity</td>
<td>( \Delta V/\Delta P )</td>
<td>0.2</td>
<td>mV/kPa</td>
</tr>
<tr>
<td>Linearity</td>
<td>( \pm 0.5% )</td>
<td>FS</td>
<td></td>
</tr>
<tr>
<td>Hysteresis</td>
<td>( \pm 0.5% )</td>
<td>FS</td>
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</tr>
<tr>
<td>Temperature effect on FS (0 to 50°C, Tref=25°C)</td>
<td>T_FS</td>
<td>±2%</td>
<td>FS</td>
</tr>
<tr>
<td>Temperature effect on Offset (0 to 50°C, Tref=25°C)</td>
<td>T_OFF</td>
<td>±1</td>
<td>mV</td>
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Error calculation:

\[ \text{Err}_{T,FS} = \pm 2 \cdot \frac{200}{100} = \pm 4 \text{kPa} \]

\[ \text{Err}_{T,OFF} = \pm 1 \div 0.2 = \pm 5 \text{kPa} \]

Repeatability

- Distribution of successive measurements of \( y \) under the same conditions

\[ \text{Repeatability} \text{\,(\%)} = \frac{\pm \Delta y_{\text{max}}}{FS} \]
Stability

- The ability to maintain a response $y$ for a constant $x$ and during a given time period

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<td>Voff</td>
<td>±1</td>
<td>mV</td>
</tr>
<tr>
<td>Sensitivity</td>
<td>ΔV/ΔP</td>
<td>0.2</td>
<td>mV/kPa</td>
</tr>
<tr>
<td>Offset Stability</td>
<td>±0.5%</td>
<td>FS</td>
<td></td>
</tr>
</tbody>
</table>

\[ Err_{Stab} = \pm 0.5 \cdot 200 / 100 = \pm 1 \text{kPa} \]

Maximal and probable error

- Maximal error

\[ Error_{\text{max}} = \pm \sum_i |\Delta x_i| \]

- Probable error

\[ Error_{\text{probable}} = \pm \sqrt{\sum_i \Delta x_i^2} = \pm \sigma \]

65% probability the actual error is within $[-\sigma, +\sigma]$
98% probability the actual error is within $[-2\sigma, +2\sigma]$
99% probability the actual error is within $[-3\sigma, +3\sigma]$
Resolution and threshold

- Resolution – the smallest detectable change of the input value
- Threshold – resolution at the origin (input = 0)

Transfer characteristics of conditioning circuits

- A measurement system is not just the sensor but the entire measurement chain
- In order to determine the global transfer characteristics, one must take into account also the transfer characteristics of conditioning circuits
- Achieving the highest sensitivity of the circuit is a common design goal
Example: voltage divider

\[ U_{out} = U_0 \frac{R_s}{R_s + R} \]

\[ U_{out} + \Delta U_{out} = U_0 \frac{R_s + \Delta R_s}{R_s + R} = U_0 \frac{R_s + R + \Delta R_s}{R_s + R} = U_0 \frac{R_s}{R_s + R} \left( 1 + \frac{\Delta R_s}{R_s + R} \right) \]

For \( \Delta R_s << R + R_s \)

\[ U_{out} + \Delta U_{out} = U_0 \frac{R_s + \Delta R_s}{R_s + R} \left( 1 - \frac{\Delta R_s}{R_s + R} \right) U_0 \frac{R_s}{R_s + R} = \frac{U_0 R}{(R_s + R)^2} \Delta R_s \]

Max sensitivity for \( R = R_s \). In that case \( \Delta U_{out} = U_0 \frac{\Delta R_s}{4R} \)

Example: Wheatstone bridge

\[ U_{out} = \frac{1}{2} U_0 - \frac{R}{2R + \Delta R} U_0 \]

\[ = \frac{\Delta R}{4(R + \frac{\Delta R}{2})} U_0 \]

\[ \approx \frac{\Delta R}{4R} U_0 \]

- No power supply noise for a balanced bridge
- The effect of temperature can be compensated by choosing resistors with the same temperature coefficient as the sensor
Force sensor (repeated from Ch 1)

- Based on a strain sensor attached to a test object

\[ \frac{\Delta l}{l} = \frac{F}{A \cdot Y} \]

Compressing \( \Delta l / l < 0 \)

\[ \frac{\Delta R}{R} = K \frac{\Delta l}{l} \]

\[ \frac{\Delta U}{U} = \frac{\Delta R}{R} \frac{I}{I} \]

Wheatstone bridge – sensitivity optimisation

\[ U_{out} \approx \frac{\Delta R}{R} U_0 \]

\[ U_{out} \approx \frac{\Delta R}{4R} U_0 \]

\[ U_{out} \approx \frac{\Delta R}{R} U_0 \]

\[ U_{out} \approx \frac{\Delta R}{2R} U_0 \]
Dynamic transfer characteristics

- Identification of the system order
- Calculating sensitivity
- Time-dependent response
- Frequency-dependent response

\[ a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = x(t) \]

- **Order 0:** \( y(t) = S \cdot x(t) \)  
  \( S \): sensitivity
- **1\(^{st}\) order:** \( \tau \frac{dy}{dt} + y = S \cdot x \)  
  \( \tau \): time constant
- **2\(^{nd}\) order:** \( \frac{1}{\omega_0^2} \frac{d^2 y}{dt^2} + \frac{2\xi}{\omega_0} \frac{dy}{dt} + y = S \cdot x \)  
  \( \omega_0 \): undamped frequency  
  \( \xi \): damping factor

A time-dependent input will result in a time-dependent output. This can be described by an ordinary differential equation.
Zero-order system: potentiometer

The model neglects:
- induced EMF in the output loop \((dx/dt)\)
- inductance of the potentiometer
- mechanical properties: mass of the sliding part, friction

\[ U_{out} = \frac{R_x}{R} U_0 = \frac{\rho \cdot \ell_x / A}{\rho \cdot \ell / A} U_0 = \frac{\ell_x}{\ell} U_0 = S \cdot \ell_x \]

\( \ell_x \): length of section \( x \)
\( \ell \): total length
\( A \): wire cross-section

1st order system

\[ \tau \frac{dy}{dt} + y = S \cdot x \]
\( \tau \): time constant

- Example: temperature measurements

Heating power for object \( \rightarrow \) sensor

\[ P_x = G_{xm} (T_x - T_m) \]

Heating power for surroundings \( \rightarrow \) sensor

\[ P_a = G_{am} (T_a - T_m) \]
1st order system

\[ \frac{dQ_m}{dt} = C \frac{dT_m}{dt} \]

\( C \): heat capacity of the sensor

\( Q_m \): amount of heat received by the sensor

\[ P_a + P_x = \frac{dQ_m}{dt} \]

\[ G_{am} (T_a - T_m) + G_{xm} (T_x - T_m) = C \frac{dT_m}{dt} \]

\[ \frac{d}{dt} \left( \frac{C}{G_{am} + G_{xm}} \right) + T_m = \frac{G_{xm} T_x}{G_{am} + G_{xm}} + \frac{G_{am} T_a}{G_{am} + G_{xm}} \]

Interfering input

\[ \tau = \frac{C}{G_{xm} + G_{am}} \]

\[ S = \frac{G_{xm}}{G_{am} + G_{xm}} \]

Solution of the equation:

\[ T_m = T_{m,final} \left( 1 - e^{-t/\tau} \right) \]

1st order system

\[ T_m = T_{m,final} \left( 1 - e^{-t/\tau} \right) \]

\( T_{m,final} \) from condition \( \frac{dT_m}{dt} = 0 \)

\[ T_{m,final} = \frac{G_{am} T_a + G_{xm} T_x}{G_{am} + G_{xm}} \]

\[ \tau = \frac{C}{G_{xm} + G_{am}} \]

For \( T_{m,final} \) to be close to \( T_x \):

\( G_{xm} \): as high as possible (good thermal contact sensor/object)

\( G_{am} \): as low as possible (bad contact sensor/ambient)

For fast measurements (low \( \tau \)):

\( C \): as low as possible (small thermometer)
How can we determine $S$ and $\tau$?

\[ \tau \frac{dy}{dt} + y = S \cdot x \]

$S$: static calibration (keep $x$ constant, wait until $y$ becomes constant)

$\tau$: apply a stepwise change of input

\[ y = S \cdot x_o (1 - e^{-t/\tau}) \]

\[ \frac{-\ln(1 - \frac{y}{Sx_o})}{Sx_o} = \frac{t}{\tau} \]

$2^{nd}$ order system: accelerometer

- MEMS (micro-electromechanical system) accelerometer

ADXL202 accelerometer

Analog Devices website
2\textsuperscript{nd} order system: accelerometer

\[ h : \text{position of the test mass} \]
\[ h_0 : \text{position of the accelerometer (package)} \]
\[ y = h - h_0 \]

\[ y = \frac{1}{\omega_0^2} \frac{d^2 y}{dt^2} + \frac{2\xi}{\omega_0} \frac{dy}{dt} + y = S \cdot a \]

\[ a = -\frac{d^2 h_0}{dt^2} \]

\[ \sum F = m \ddot{a}_M \quad y = h - h_0 \]

\[ \lambda \frac{dy}{dt} + ky = -m \frac{d^2 h}{dt^2} \]

\[ m \frac{d^2 y}{dt} + \lambda \frac{dy}{dt} + ky = -m \frac{d^2 h_0}{dt^2} \]
2\textsuperscript{nd} order system

- Response to a step change of input

\[ \frac{1}{\omega_0^2} \frac{d^2 y}{dt^2} + \frac{2\xi}{\omega_0} \frac{dy}{dt} + y = S \cdot x \]

For $\xi < 1$

\[ \frac{y}{Sx_0} = 1 - \frac{e^{-\xi \omega_o t}}{\sqrt{1 - \xi^2}} \sin \left( \sqrt{1 - \xi^2} \omega_o t + \phi \right) \]

\[ \phi = \text{Arc} \sin \left( \sqrt{1 - \xi^2} \right) \]

\[ \frac{y}{Sx_o} = 1 - e^{-\xi \omega_o t} \]

\[ \frac{y}{Sx_o} = 1 - \omega_o t \cdot e^{-\omega_o t} \]

\[ \frac{y}{Sx_o} = 1 - \frac{e^{-\xi \omega_o t}}{\sqrt{1 - \xi^2}} \sin \left( \sqrt{1 - \xi^2} \omega_o t + \phi \right) \]

with $\phi = \text{Arc} \sin \left( \sqrt{1 - \xi^2} \right)$
2\text{nd} order system

- Response to pulsed change of input (pulse width $\ll$ period of oscillations)

  \begin{align*}
  \text{overdamped} & \quad \xi > 1 \\
  \frac{y}{Sx_0 \omega_o} & = \frac{e^{-\xi \omega_o t}}{\sqrt{\xi^2 - 1}} \sinh(\sqrt{\xi^2 - 1} \omega_o t) \\

  \text{critical damping} & \quad \xi = 1 \\
  \frac{y}{Sx_0 \omega_o} & = \omega_o t e^{-\omega_o t} \\

  \text{underdamped} & \quad \xi < 1 \\
  \frac{y}{Sx_0 \omega_o} & = \frac{e^{-\xi \omega_o t}}{\sqrt{1 - \xi^2}} \sin\left(\sqrt{1 - \xi^2} \omega_o t\right)
  \end{align*}

How can we determine these parameters?

- $S$ : static calibration (keep input constant, wait until output becomes constant)
- $\omega_0$ and $\xi$ : apply a step-wise change of the input

Other parameters:
- Measure $E$ (excursion) and $T$ (period)
2\textsuperscript{nd} order system – stepped input

\[
\frac{y}{Sx_0} = 1 - \frac{e^{-\xi \omega_0 t}}{\sqrt{1 - \xi^2}} \sin \left( \frac{\sqrt{1 - \xi^2} \omega_0 t + \varphi}{\frac{2\pi}{T}} \right)
\]

\[
\omega_0 = \frac{2\pi}{T \sqrt{1 - \xi^2}}
\]

Calculate $\xi$:
- find $E$ from the condition \[\frac{d}{dt} \left( \frac{y}{Sx_0} \right) = 0\]

\[
\xi = \frac{1}{\sqrt{\left( \frac{\pi}{\ln E} \right)^2 + 1}}
\]

2\textsuperscript{nd} order system – pulsed input

\[
\frac{1}{\omega_0^2} \frac{d^2 y}{dt^2} + \frac{2\xi}{\omega_0} \frac{dy}{dt} + y = S \cdot x
\]

$S$ : static calibration (keep input constant, wait until output becomes constant)
$\omega_0$ and $\xi$ : apply a pulsed input

Measure $y_0/y_i$ and $T$
2nd order system – pulsed input

\[ \frac{y}{S_x \omega_o} = \frac{e^{-\zeta \omega_o t}}{\sqrt{1-\zeta^2}} \sin \left( \sqrt{1-\zeta^2} \frac{2\pi}{T} t \right) \]

Calculating \( \zeta \): deduce \( y_i \) from the maxima of \( y \)

\[ \ln \frac{y_o}{y_m} \approx \frac{\ln y_m - \ln y_0}{2\pi m} \]

\[ Lny_m = Lny_0 - 2\pi m \zeta \]

assuming \( \sqrt{1-\zeta^2} \approx 1 \)

\[ \omega_0 = \frac{2\pi}{T \sqrt{1-\zeta^2}} \quad \text{for} \quad \zeta < 1 \]

\[ \omega_o t_o = \frac{\pi}{2} \]

\[ \omega_o t_m = \frac{\pi}{2} + 2m\pi \]

2nd order system – oscillating input

\[ \frac{1}{\omega_o^2} \frac{d^2 y}{dt^2} + \frac{2\zeta}{\omega_o} \frac{dy}{dt} + y = S \cdot x \]

\( S \): static calibration (keep input constant, wait until output becomes constant)

\( \omega_0 \) and \( \zeta \): apply an oscillating input with a variable frequency \( \omega \)

Measure the maximum

\[ \frac{|y|}{S|X|} = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_o^2}\right)^2 + 4 \zeta^2 \frac{\omega^2}{\omega_o^2}} \}} \]

|\( Y \), |\( X \)|: output and input amplitudes
2nd order system – oscillating input

\[
\frac{|Y|}{S|X|} = \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + 4\xi^2 \frac{\omega^2}{\omega_0^2}}}
\]

\[\omega = \omega_r \quad \text{for} \quad \frac{|Y|}{S|X|} = \text{max} \quad \text{or} \quad \left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + 4\xi^2 \frac{\omega^2}{\omega_0^2} = \text{min}\]

\[\omega_0 = \frac{\omega_r}{\sqrt{1 - 2\xi^2}} \]

For \(\omega \gg \omega_r\),

\[
\frac{|Y|}{S|X|} \approx \frac{1}{\sqrt{\frac{\omega^4}{\omega_0^4} + 4\xi^2 \frac{\omega^2}{\omega_0^2}}} \approx \frac{\omega_0^2}{\omega^2}
\]

\[20\log\left(\frac{|Y|}{S|X|}\right)\]

\[\omega_r\quad \text{dB}\]

Key points

• A measurement system is often modelled by a linear differential equation of the order 0, 1 or 2
• Static transfer parameters allow us to predict the measurement error
• The sensitivity and dynamic transfer parameters allow us to identify the order of the system
Chapter 3: Noise

Measurement chain

Data analysis (recording, averaging, etc.)

Acquisition (Analog – digital conversion)

Arduino UNO board

Conditioning circuit

Noise reduction

Sensor

Modeling

Conditioning circuit

$U_0 = 5 \text{ V}$

$U_{out}$

$R_{load}$

$R_{sensor}$

$0V$
Measurement chain

Chapter 1

Sensor → Conditioning → Noise reduction and signal processing → Acquisition → Action

Chapter 2

Modeling

Chapter 3

Chapter 4

Chapter 5 and 6

Noise: Example

$u_n(t)$
Noise is not always bad

Noise estimation and suppression

• Sources
• Extrinsic noise
  – Conductive coupling
  – Capacitive coupling
  – Magnetic coupling
• Noise suppression using differential measurements
  – Common mode voltage
  – Suppression of the common mode voltage
  – Instrumentation amplifier
• Intrinsic noise
  – Thermal noise
  – Shot noise
  – 1/f noise
  – Noise estimation
Noise sources

Extrinsic noise \( \text{Noise}_2(t) \)
- External influence
  - Electrical
  - Magnetic
  - Electromagnetic
  - Mechanical (vibration, sound)
  - Thermal (temperature variation)

Intrinsic noise \( \text{Noise}_1(t) \)
- Internal to the circuit and device
  - Thermal
  - Shot noise
  - 1/f

Extrinsic noise: perturbations

- Coupling mechanisms
  - Conductive (galvanic) coupling
  - Capacitive (electrostatic) coupling
  - Magnetic coupling

- Coupling modes
  - Common mode
  - Differential mode
Conductive coupling

- Power supplies and the ground

Reference connections of a circuit

- Power ground
  - sometimes also referred to as simply “ground”, Earth
  - the Earth is considered to be a perfect conductor and a sink for charge
  - ground potential is by reference 0V
  - connection provided by the power supply line or a dedicated line
Reference connections of a circuit

• Signal ground
  - voltage reference against which all the voltages on the input/output terminals are measured against
  - may be connected to the power ground, usually through the chassis, either directly or through a ~1MOhm resistor
  - signal ground that is not connected to the power ground is called a “floating ground”.

Example: Oscilloscope
Example: battery-powered voltmeter

Example of connections

\[ R_1 \]
\[ R_2 \]
Conductive coupling – ground loop

- Cause: finite resistance of connecting wires
- Influence of the signal ground potential difference
- Occurs also if the signal ground is connected to the ground

\[ u_{in} = u_s + u_G \]

\[ u_G = Z_G \cdot i_G \]
\[ Z_G = R_G + jL_G \omega \]

Reducing the conductive coupling

- Reducing ground connections: star grounding
Connecting instruments

Power supply, 230V, 50Hz

- Oscilloscope
  - Phase
  - Neutral
  - Ground

- Voltmeter
  - Voltmeter
  - Signal ground

- Power supply, 230V, 50Hz
  - Phase
  - Neutral
  - Ground

- Generator
  - Referenced

- Voltage source
  - Signal ground

- Battery
  - Floating

Measuring instruments

Sources

Referenced and floating sources and amplifiers

- Referenced source (with respect to the signal or power ground)
- Floating source (isolated from the signal or power ground)
- Asymmetric referenced amplifier (with respect to the power ground)
- Asymmetric floating amplifier (isolated from the power ground)
Referenced source – floating amplifier

- Amplifier with a non-referenced single-ended input (NRSE)

Floating source – referenced amplifier

- Amplifier with a referenced single-ended input (RSE)
Referenced source – referenced amplifier

- Results in a ground loop
- Avoid

\[ u_{in} = u_s + u_G \]

Complex impedances in an ac circuit

<table>
<thead>
<tr>
<th></th>
<th>Resistor</th>
<th>Capacitor</th>
<th>Inductor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Impedance ( Z )</td>
<td>( Z_R = R )</td>
<td>( Z_C = \frac{1}{jC\omega} )</td>
<td>( Z_L = jL\omega )</td>
</tr>
<tr>
<td>Differential equation</td>
<td>( v = iR )</td>
<td>( i = C \frac{dv}{dt} )</td>
<td>( v = L \frac{di}{dt} )</td>
</tr>
<tr>
<td>Phase difference (( i ) with respect to ( v ))</td>
<td>0</td>
<td>+90° (( i ) ahead of ( v ))</td>
<td>-90° (( i ) lagging behind ( v ))</td>
</tr>
</tbody>
</table>

Ohm’s Law: \( V = I \cdot Z_{total} \) \( V \) and \( I \) are also complex numbers (amplitude and phase)

1. Apply regular Kirchhoff rules, calculate \( Z_{total} \) according to rules for parallel and serial addition of resistors
2. Keep complex numbers until the end
3. Calculate absolute values and phase (if interested in it)
Capacitive coupling

- Capacitive coupling between the measurement circuit and a noise source (for example the 220 V powerline network)

\[ i_c \approx C_{ef} \frac{dU_n}{dt} \]

\( C_{ef} \) – typically 0.1-1000 pF

---

Capacitive coupling - example

\[ u_n = 230 \cos(2 \pi \cdot 50 \cdot t) \]

\[ i_c \approx C_{ef} \frac{du_n}{dt} \]

\( C_{ef} = 1.4 \, \text{pF} \)
\[ Z_s + Z_{cable,1} + Z_{cable,2} = 10 \, \Omega \]
\[ Z_{in} = 10 \, \text{MOhm} \]

\[ \downarrow \quad u_A = \frac{Z_{in}}{Z_{cable,1} + Z_{cable,2} + Z_{in} + Z_s} u_s + u_{c,A} \]

\[ u_{c,A} = (Z_{cable,1} + Z_{cable,2} + Z_s) i_c \approx 1 \mu V \]
Shielding

Without Shielding

With Shielding
How to connect the shield

• Needs to be connected to the ground or the reference

![Good](image1.png) ![Bad](image2.png)

Ground loop

$u_2$ capacitively couples into the signal

Magnetic coupling

• Magnetic field generated by one circuit induces EMF in the other (measurement) circuit

Physical representation

\[ \phi = \int_{\text{measurement circuit area}} \bar{B}d\bar{A} = Mi_n \]

$A$: area of the measurement (receiving) circuit

$M$: mutual inductance

Electromotive force

\[ \mathcal{E}_i = -\frac{d\phi}{dt} \]

\[ u_{\text{induced}} = M \frac{di_n}{dt} \]
Protection: twisted pairs

- Induced EMF in each twist

\[ \varepsilon_i = -\frac{\partial}{\partial t} \int_{s_i} B \cdot d\mathbf{S} \]

Opposite signs of EMF in neighbouring twists

Magnetic coupling to ground loops

- External time-varying magnetic field \( B \) induces a voltage \( u_i \)

\[ u_i \approx A \frac{dB}{dt} \quad A \text{ - loop area} \]

\[ u_{in} = u_s + u_i \]
Magnetic shielding

Without shielding

\[ i_p \]

Grounded shielding

\[ i_p \]

Voltage induced in 2 by 1:

\[ u_{2,0} = j \omega M_{12} i_p \]

Voltage drops over the shield (3):

\[ i_3 (R_3 + j \omega L_3) = -j \omega M_{13} i_p \]

Voltage drops over the inner conductor (2):

\[ u_2 = j \omega M_{12} i_p + j \omega M_{23} i_3 \]

Effectiveness of shielding depends on the frequency

\[ \left| \frac{u_2}{u_{2,0}} \right|^2 = \frac{1}{1 + \left( \frac{f}{f_c} \right)^2} \]
Magnetic shielding

\[ k = 10 \log \left| \frac{u_2}{u_{2,0}} \right|^2 = 10 \log \frac{1}{1 + \left( \frac{f}{f_c} \right)^2} \]

Induced voltage

\[ u_2 = \frac{1}{1 + j \left( \frac{f}{f_c} \right)} \left( u_{2,0} = \frac{j \omega M_{12} i_p}{1 + j \left( \frac{f}{f_c} \right)} \right) \]

\[ |u_2| = \frac{2\pi f M_{12} |i_p|}{\sqrt{1 + \left( \frac{f}{f_c} \right)^2}} \]

For \( f \gg \)

\[ |u_2| \approx \omega_c M_{12} |i_p| \quad \omega_c = \frac{R_3}{L_3} \]

- Shielding not effective for \( f < f_c \), however induced voltage is also small there
- For \( f >> \), \( u_2 \) depends on \( M \) and \( \omega_c \)
- We should:
  - Decrease \( M_{12} \)
  - Decrease \( \omega_c \) (decrease \( R_3 \), increase \( L_3 \))

Noise estimation and suppression

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  - Shot noise
  - 1/f noise
  - Noise estimation
Common and differential mode signals and perturbations

3.35

Common mode perturbation

Differential mode perturbation

Common mode parasitic voltage

\[ u_{cm,p} = \frac{u_{p1} + u_{p2}}{2} \]

Differential mode parasitic voltage

\[ u_{diff,p} = u_{p1} - u_{p2} \]

Common mode voltage - example

- Common mode voltage \( u_{cm} \): voltage common to \( u_A \) and \( u_B \) that does not carry a useful signal

\[ u_{cm} = \frac{u_A + u_B}{2} \]

\[ u_s = u_A - u_B \]
Common mode voltage - example

Example: \( u_s = 10 \text{V}, \Delta R/R = 0.01 \)
\( u_m = 25\text{mV} \) and \( u_{cm} = 5\text{V} \gg u_m \)

\( u_{cm} \) can be much larger than \( u_m \)

Common mode voltage - example

- Common mode voltage from a ground loop

\( R_{in} : \) large (open circuit, no current through \( R_s \))

\[ u_A = u_s + u_G \]
\[ u_B = u_G \]
\[ u_{cm} \approx u_G \text{ for } u_s \ll u_G \]

For the case of an ideal differential amplifier:

\[ u_{input} = u_A - u_B = u_s \]
Differential measurements – shielded cables
Differential measurements – shielded cables

Usually:

\[ R_{in} \gg R_s, R_c, R_{shield}, R_G \]

Typical values:

\[ R_{in} = 10 \text{ M\ensuremath{\Omega}}, \quad R_s = 1 \text{ k\ensuremath{\Omega}}, \]
\[ R_c = R_{shield} = 10 \text{ \ensuremath{\Omega}}, \]
\[ R_G = 1 \text{ \ensuremath{\Omega}} \]

\[ \Delta u_{in,G} = u_{in,G}^+ - u_{in,G}^- = \frac{R_{in}}{2} (i^+ - i^-) \]

Huge reduction in noise!
(A more realistic) Differential amplifier

\[ u_o = A_2 u_+ - A_1 u_- \]

\[ u_{cm} = \frac{u_+ + u_-}{2} \quad u_d = u_+ - u_- \]

\[ u_o = \frac{A_1 + A_2}{2} u_d + \left( \frac{A_2 - A_1}{A_{cm}} \right) u_{cm} \]

\[ A_d : \text{differential gain} \]
\[ A_{cm} : \text{common mode gain} \]

For an ideal amplifier:
\[ A_2 = A_1 \]

Common mode rejection ratio (CMRR)

\[ CMRR = \frac{A_d}{A_{cm}} \]

\[ CMRR_{dB} = 20 \log \left| \frac{A_d}{A_{cm}} \right|, dB \]

Example:
Standard: cca 90dB CMRR
For \( u_{cm} = 10V \rightarrow \pm 316 \mu V \) on the output

Output voltage:

\[ u_{out} = A_d \cdot u_d + A_{cm} \cdot u_{cm} \]

\[ u_{out} = A_d \left( u_d + \frac{1}{CMRR} u_{cm} \right) \]

How to measure \( A_{cm} \):

\[ A_{cm} = \frac{u_{out}}{u_{cm}} \]
**Instrumentation amplifier**

- High-performance differential amplifier
  - High CMRR: 100 db (at 50 Hz)
  - Low input current (nA or pA)
  - Low output impedance: 0.1 $\Omega$
  - Large input impedance: $10^{10}$ $\Omega$
  - Large temperature stability
  - Programmable differential gain

**Isolation amplifier**

- Signal is transmitted through magnetic (transformer), optical (LED – photodiode pair) or capacitive (capacitor) coupling

  - Recommended for $u_{cm} > 70\%$ of the power supply voltage
Isolation amplifier

- Decreases high levels of common mode voltage by breaking ground loops
- Protects the source from voltage surges in the instrument – important for medical applications ("source" is usually a person)

Isolation mode rejection ratio

- $u_{cm1} : 10$s of volts
- $u_{cm2} : \text{can be } >1000 \text{ V}$

$$u_{out} = A_d \left( u_d + \frac{1}{CMRR} u_{cm1} + \frac{1}{IMRR} u_{cm2} \right)$$

- CMRR: common mode rejection ratio ($>100 \text{ db – factor } 10^5$)
- IMRR: isolation mode rejection ratio ($>140 \text{ db – factor } 10^7$)
- $A_d : \text{differential gain}$
Key points

• Common mode voltage is a signal (sometimes noise, sometimes just a voltage offset) present on both inputs of the amplifier
• In case it corresponds to noise, we would of course like to eliminate it
• In case it corresponds to an offset, we would like to avoid it saturating or destroying the amplifier
• Differential, instrumentation and isolation amplifiers reduce the effect of common mode voltage

Noise estimation and suppression

• Sources
  • Extrinsic noise
    - Conductive coupling
    - Capacitive coupling
    - Magnetic coupling
  • Noise suppression using differential measurements
    - Common mode voltage
    - Suppression of the common mode voltage
    - Instrumentation amplifier
• Intrinsic noise
  - Thermal noise
  - Shot noise
  - 1/f noise
  - Noise estimation
Intrinsic noise

- **Stationary:**
  - All statistical parameters (average, standard deviation, etc) are time-independent
  - Noise estimated during one interval of time $\Delta T_1$ is the same as in another interval $\Delta T_2$

- **Ergodic**
  - Time average is the same as the ensemble average

- **Stationary ergodic noise:** a stationary noise for which the probability that the noise voltage lies within any given interval at any time is nearly equal to the fraction of time that the noise voltage lies within this interval if a sufficiently long observation interval is recorded
Ergodicity

1. Take N resistors
2. Measure their resistance vs. time -> N traces of R vs. time
3. Calculate average over time for the trace of any one resistor: time average
4. Now look at all the N traces and choose points from each trace corresponding to the same time $t_x$
5. Calculate the average -> ensemble average

Time average should have the same value as the ensemble average

Characteristics of noise

$u(t)$ – voltage measurement as a function of time

$u_n(t) = u(t) - \bar{u}$

$u_n(t)$ – noise

$u_n(t)$ and $i_n(t)$ – voltage or current noise, deviation from the average
### Characteristics of noise

<table>
<thead>
<tr>
<th>Characteristics of noise</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Mean value</strong> (of the deviation)</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td><strong>Mean square of the deviation</strong> (variance)</td>
</tr>
</tbody>
</table>

### Characteristics of noise

<table>
<thead>
<tr>
<th>Characteristics of noise</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Effective value</strong> (DC voltage that would give you the same power dissipation as the noise)</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td><strong>Average power</strong></td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>
Reminder: frequency spectrum

- Sinusoidal signals

\[ u = A \sin \left( \frac{2\pi}{T_o} t \right) + B \sin \left( \frac{2\pi}{T_1} t \right) \]

Time domain

Frequency domain

\[ f_o = \frac{1}{T_o} = 1kHz \]
\[ f_1 = \frac{1}{T_1} = 1.5kHz \]

Generalisation

- Periodic signals

- Arbitrary signals

Continuous spectrum
Noise spectral density

- Voltage or current noise is a superposition of periodic noise with a spectrum ranging from 0 to infinity.
- The noise amplitude depends on the frequency bandwidth: \( f_{max} - f_{min} \).
- Noise spectral density
  \[
  \Phi_{u,n}(f) = \frac{d(u_n^2)}{df} \quad \text{V}^2/\text{Hz} \\
  \Phi_{i,n}(f) = \frac{d(i_n^2)}{df} \quad \text{A}^2/\text{Hz}
  \]
- Allows us to express the electrical power due to noise in the measurement bandwidth.

Voltage (Current) noise mean square

\[
\overline{u_n^2} = \int_{f_{min}}^{f_{max}} \Phi_{u,n}(f) df; \quad \overline{i_n^2} = \int_{f_{min}}^{f_{max}} \Phi_{i,n}(f) df;
\]

For spectral density independent of frequency:

\[
\overline{u_n^2} = \Phi_{u,n} \cdot (f_{max} - f_{min})
\]

Summary

<table>
<thead>
<tr>
<th>Instantaneous values</th>
<th>( u_n(t) )</th>
<th>( i_n(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Effective values</td>
<td>( U_{n,\text{eff}} )</td>
<td>( I_{n,\text{eff}} )</td>
</tr>
<tr>
<td>Average power</td>
<td>( P_n = \frac{\overline{u_n^2}}{R} = \frac{U_{n,\text{eff}}^2}{R} )</td>
<td>( P_n = R\overline{i_n^2} = RI_{n,\text{eff}}^2 )</td>
</tr>
<tr>
<td>Spectral densities</td>
<td>( \Phi_{u,n}, \text{V}^2/\text{Hz} )</td>
<td>( \Phi_{i,n}, \text{A}^2/\text{Hz} )</td>
</tr>
<tr>
<td></td>
<td>( \sqrt{\Phi_{u,n}}, \text{V}/\sqrt{\text{Hz}} )</td>
<td>( \sqrt{\Phi_{i,n}}, \text{A}/\sqrt{\text{Hz}} )</td>
</tr>
<tr>
<td>Estimation of noise intensity (noise indep. of frequency)</td>
<td>( U_{n,\text{eff}} = \sqrt{\Phi_{u,n} \cdot (f_{max} - f_{min})} )</td>
<td>( I_{n,\text{eff}} = \sqrt{\Phi_{i,n} \cdot (f_{max} - f_{min})} )</td>
</tr>
</tbody>
</table>
Example

The noise for amplifier LF356 in the frequency band [0-40Hz]:

\[ \frac{U_{n,\text{eff}}}{\sqrt{\Delta f}} = 15nV / \sqrt{\text{Hz}} \]

Effective value: \[ U_{n,\text{eff}} = 15nV \times \sqrt{40} = 94nV \]

Signal to noise ratio (SNR)

\[ SNR = \frac{\text{Signal power}}{\text{Noise power}} = \frac{s}{n} \]

\[ SNR_{\text{dB}} = 10\log \frac{s}{n} \]

• Example: signal \( U_{\text{eff}} = 10 \text{ mV} \), noise \( U_{n,\text{eff}} = 4.9 \text{ mV} \)

\[ SNR_{\text{dB}} = 20\log \frac{U_{\text{eff}}}{U_{n,\text{eff}}} = 20\log \frac{10}{4.9} = 6.2\text{dB} \]
Types of intrinsic noise

- **Thermal noise (Johnson noise)**
  - Fluctuations in resistance due to random motion of atoms that influence conduction electrons
  - White noise (same for all frequencies)
- **Shot noise**
  - Fluctuations in current due to the fact that the current is composed of discrete charge carriers
  - White noise
- **1/f noise (flicker noise, pink noise)**
  - Fluctuations in resistance due to instabilities in contacts, atom migration, impurities in the conductive channel

**White noise**

- $\Phi_n(f)$ does not depend on $f$
- In practice, we consider the white noise in a limited frequency band

\[\Phi_n(f) = \text{const} \quad -\infty < f < +\infty\]
Thermal noise

- Power spectral density of voltage variance (i.e. voltage variance per Hz of bandwidth) across a resistor at finite temperature $T$:

$$\Phi_{u,n} = 4k_B TR$$

- Boltzmann constant, $k_B = 1.38 \times 10^{-23}$ J/K
- $T$ : temperature in Kelvins
- $R$ : Resistance
- $\Delta f$ : Bandwidth

Example: 50Ω resistor at room temperature (300 K), 1 Hz bandwidth

$$U_{n,eff} = \sqrt{\Phi_{u,n} \cdot \Delta f} = \sqrt{4k_B TR \cdot \Delta f} = 1\text{nV}$$

- A resistor in short circuit dissipates a noise power of:

$$P = \frac{U_{n,eff}^2}{R} = 4k_B T \Delta f$$

Independent of $R$!

Equivalent circuits

Noisy resistor at temperature $T$

\[=\]

Noisless resistor

Noise voltage

$U_{n,eff}^2 / \Delta f = 4k_B TR$

Thevenin equivalent circuit

or

Noisless resistor

Noise current

$I_{n,eff}^2 / \Delta f = 4k_B T / R$

Norton equivalent circuit
Maximal dissipated noise

• Noise delivered to $R_L$ from $R_n$ : $P_{n,L}$

\[
P_{n,L} = I_{n,\text{eff}}^2 R_L \quad I_{n,\text{eff}} = \frac{U_{n,\text{eff}}}{R_n + R_L}
\]

\[
P_{n,L} = \frac{U_{n,\text{eff}}^2 R_L}{(R_n + R_L)^2}
\]

$P_{n,L} = \text{max for } R_n = R_L$

\[
P_{n,L} = \frac{U_{n,\text{eff}}^2}{4R_n} = k_B T \Delta f
\]

Independent of $R$!

Shot noise

• Schottky noise

• Due to fluctuations in the number of charge carriers, described by Poisson distribution

\[
SNR = \frac{N}{\Delta N} = \frac{N}{\sqrt{N}} = \sqrt{N}
\]

\[
I_{n,\text{eff}} = \sqrt{2ei\Delta f}
\]

$i$ : average current flowing through the circuit

$e$ : elementary charge ($1.6 \times 10^{-19}$ C)
1/f noise

• In general dominant under 500 Hz

\[ \Phi_n \sim \frac{1}{f^\alpha} \quad 0.8 < \alpha < 1.3 \quad \text{For } \alpha = 1: \quad \Phi_n = \frac{K}{f} \]

• Noise in the frequency band \([f_{\text{min}}, f_{\text{max}}]\):

\[
\int_{f_{\text{min}}}^{f_{\text{max}}} \Phi_n(f) df = K \ln \frac{f_{\text{max}}}{f_{\text{min}}}
\]

Same dissipated power between 1-10 Hz and 0.1-1 Hz!

Total noise due to multiple sources

• Different noise sources can be considered to be independent
• The total dissipated power is the sum of dissipated powers from each source
• Squares of current or voltage are summed up

\[
u_n^2 = u_{n1}^2 + u_{n2}^2 + i_{n3}^2 R_{n3}^2 + ... + u_{nn}^2
\]
\[
i_n^2 = i_{n1}^2 + i_{n2}^2 + u_{n3}^2 / R_{n3}^2 + ... + i_{nn}^2
\]
Signal and noise

Noise at different frequencies

• Low frequencies: $1/f$ dominant

• High frequencies: white noise
  - Thermal noise, shot noise

• Total noise depends on the frequency
  - High for DC measurements, better in white noise region

• Problem: low-frequency and DC measurements

Noise reduction

• Limiting the bandwidth
• Filtering
• Averaging
Key points

• **Extrinsic noise** is most often due to galvanic, electrostatic or magnetic coupling
• **Ground loops** induce noise in measurement circuits
• **Common mode voltage** appears on both inputs to an amplifier
• **Differential and isolation amplifiers** reduce the effect of common mode voltage
• Common mode voltage can be reduced by **shielding**
• Noise coming from the system itself is **intrinsic**: thermal, shot and 1/f
• Equivalent noise power is calculate from the **power spectral density** (V²/Hz) by taking into account the bandwidth
• Intrinsic noise can be reduced by filtering, averaging,...
Chapter 4: Data Acquisition

Measurement chain

Data analysis (recording, averaging, etc.)

Acquisition (Analog – digital conversion)

Arduino UNO board

Conditioning circuit

Noise reduction

Conditioning

Sensor

Modeling

$U_0 = 5 \text{ V}$
Measurement chain

Chapter 1

Sensor → Conditioning → Noise reduction and signal processing → Acquisition → Action

Chapter 2

Modeling

Chapter 3

Chapter 4

Chapter 5 and 6

Analog signal

• Continuous both in amplitude and time and can assume an infinite number of different values – infinite resolution
Analog music recording

• Late 1980’s, early 90s

Digital signal

• Signal is represented as two values ("low" and "high"), with distinct voltage levels
**Bit and byte**

- A digital signal can represent either a state of a quantity (bit) or be an element of a unit of information (byte)

![Bit representation](image)

![Byte representation](image)

**Possible values**

- Bit: 5 V or 0 V
- Byte (composed of N bits): 5 V or 0 V

**Analog – digital conversion**

- The digital signal is:
  - Less perturbed by noise
  - Easier to process, transmit or store
- Signal is often converted between analog – digital forms
  - Music playback, generation of analog voltages using computer-controlled instruments etc.
- AD and DA converters
Sampling

• Before the conversion, the analog signal is sampled
• The signal to be sampled is multiplied with a pulse train signal

What is the best choice for the sampling frequency?
Reminder: frequency spectrum

- Sinusoidal signals

Time domain

\[ u = A \sin(\frac{2\pi}{T_o}t) + B \sin(\frac{2\pi}{T_1}t) \]

Frequency domain

\[ f_o = \frac{1}{T_o} = 1kHz \]

\[ f_1 = \frac{1}{T_1} = 1.5kHz \]

Representation

- Periodic signal

\[ x(t) = A \left[ \frac{4}{\pi} \sin \omega_o t + \frac{\sin 3\omega_o t}{3} + \frac{\sin 5\omega_o t}{5} + \ldots \right] \]

- Non-periodic signal

\[ M(f) \]
Reconstruction of a square signal

Multiplication operation

- Product:
  \[ \cos(2\pi f_1 t) \cdot \cos(2\pi f_2 t) = \frac{\cos(2\pi (f_2 - f_1) t) + \cos(2\pi (f_2 + f_1) t)}{2} \]

Signal A \times B
Sampling of a periodic signal

Time domain

\[ x(t) = A \cdot \sin \omega t \]

\[ f_1 = \frac{\omega_1}{2\pi} \]

Frequency domain

\[ f \]

\[ f_1 \]

Sampling period, 0.2s

Sampling of an arbitrary signal

Time domain

Frequency domain

\[ f_{\text{max}} \]

\[ f_s \]

\[ f_{\text{max}} \]

\[ f_s \]

Frequency domain

\[ f_{\text{max}} \]

\[ f_s - f_{\text{max}} \]

\[ f_s f_{\text{max}} \]
Choice of the sampling frequency

Analog signal

Good sampling

Bad sampling

Example: sinusoidal signal, frequency $f_0$

Original signal + sampling points

Recovered signal

$T = 1/F$

$T = 1/f_s$

Original waveform

Samples

Reconstructed waveforms

Original waveform

Reconstructed waveform

Original signal + sampling points

Recovered signal

$T = 1/f_s$

Original waveform

Reconstructed waveform

Original signal + sampling points

Recovered signal

$T = 1/f_s$

Original waveform

Reconstructed waveform

Original signal + sampling points

Recovered signal

$T = 1/f_s$

Original waveform

Reconstructed waveform

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Original signal + sampling points

Recovered signal

$T = 1/f_s$

Original waveform

Reconstructed waveform

Original signal + sampling points

Recovered signal

$T = 1/f_s$

Original waveform

Reconstructed waveform

Original signal + sampling points

Recovered signal

$T = 1/f_s$

Original waveform

Reconstructed waveform

Original signal + sampling points

Recovered signal

$T = 1/f_s$
Example: fixed sampling frequency

Good sampling

Original signal

Sampled signal

Reconstructed signal

Bad sampling

Nyquist - Shannon theorem of sampling

\[ f_s > 2f_{\text{max}} \]

Reconstruction filter

In practice, \( f_s \) several times larger than \( f_{\text{max}} \)
Spectral folding

Good $f_s$

Bad $f_s$

Distorted signal

Antialiasing filter

Without filter

With filter

Analog signal $\rightarrow$ Antialiasing filter $\rightarrow$ A/D

Eliminates unwanted frequencies ($<f_s/2$) before sampling
### Decimal – binary number conversion

- **Decimal system**
  \[ 1234_{10} = (1 \times 10^{3}) + (2 \times 10^{2}) + (3 \times 10^{1}) + (4 \times 10^{0}) \]

- **Binary system**
  \[ 1101_{2} = (1 \times 2^{3}) + (1 \times 2^{2}) + (0 \times 2^{1}) + (1 \times 2^{0}) \]

#### Conversion binary -> decimal

\[ 11010_{2} = (1 \times 2^{4}) + (1 \times 2^{3}) + (0 \times 2^{2}) + (1 \times 2^{1}) + (0 \times 2^{0}) \]

\[ = 16 \quad + \quad 8 \quad + \quad 0 \quad + \quad 2 \quad + \quad 0 \]

\[ = 26_{10} \]
Conversion decimal -> binary number

- **Decimal to binary number**

<table>
<thead>
<tr>
<th>26₁₀</th>
<th>quotient</th>
<th>remainder</th>
</tr>
</thead>
<tbody>
<tr>
<td>26</td>
<td>13</td>
<td>0</td>
</tr>
<tr>
<td>13</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

read the number from starting from the last digit
=11010

**Encoding**

Continuously changing variable

→ Digital form (binary code)

```
3 → 0011₂ = 0×2³ + 0×2² + 1×2¹ + 1×2⁰ = 2 + 1 = 3₁₀
```

Conversion of a decimal number \( N_{\text{dec}} \) into a binary code

\[
N_{\text{dec}} \rightarrow a_1a_2a_3\ldots a_{n-1}a_n
\]

\[
N_{\text{dec}} = \sum_{i=1}^{n} a_i2^{i-1} = a_12^{n-1} + a_22^{n-2} + \cdots + a_{n-1}2^1 + a_n2^0
\]

\[
= 2^n \sum_{i=1}^{n} a_i 2^{-i} = 2^n (a_1 2^{-1} + a_2 2^{-2} + \cdots + a_n 2^{-n})
\]

\( a_1 \) MSB – most significant bit

\( a_n \) LSB – least significant bit
Quantisation

ADC value (analog to digital converter output value) – ranges from 0 to $2^n$

$n$: number of bits

code($N_{dec}$): $a_1a_2a_3...a_{n-1}a_n$

Example: $n=3$ bit

- Convert 4.5V with an 8-bit AD converter with a FS = 5V

  \[
  N_{dec} = 256 \times \frac{4.5}{5} = 230 = (11100110)_2
  \]

  \[
  \text{Resolution} = \frac{5}{256} = 0.019V \quad (0.01953V)
  \]

- Convert an ADC value of 156 to volts (8 bit converter and FS = 5V)

  \[
  U_D = \frac{N_{dec}}{2^n} \times FS = \frac{156}{256} \times 5 = 3.0469V \quad U_{in} = 3.0469 \pm 0.0098V
  \]
Quantisation error

Quantisation error = \(|U_D - U_{in}|\)

= \(\left| \frac{N_{dec}}{2^n} \cdot FS - U_{in} \right|\)

Max quantisation error = \(\pm \frac{0.5 \cdot FS}{2^n} = \pm \frac{q}{2}\)

Quantisation error as noise

Power of the noise associated with the quantisation error \((R = 1\Omega)\)

\[ P_n = \frac{1}{T} \int_0^T u_n^2(t) dt = \frac{2}{T_s} \int_0^{T_s/2} \left( \frac{q/2}{T_s/2} t \right)^2 dt = \]

\[ = \frac{2}{T_s} \frac{q^2}{T_s^2} \left[ \frac{t^3}{3} \right]_0^{T_s/2} = \frac{2}{T_s^3} \frac{T_s^3}{24} = \frac{q^2}{12} \]
Resolution

• The smallest detectable variation of the input

\[ \text{Resolution} = \frac{1}{2^n} FS = q \]

Example:

FS = 5V, n = 4
resolution = 5V/2^4 = 0.31V

Example

• Convert 4.5V with an 8-bit AD converter with a FS = 5V

\[ N_{dec} = 256 \times \frac{4.5}{5} = 230 = (11100110)_{2} \]

\[ Error = \left| \frac{230}{256} \times 5 - 4.5 \right| = \left| 4.4922 - 4.5 \right| = 0.0078V \]

\[ \text{Max error} = \frac{0.5 \times 5}{256} = 0.0098V \]

\[ \text{Resolution} = \frac{5}{256} = 0.019V \quad (0.01953V) \]

• Convert an ADC value of 156 to volts (8 bit converter and FS = 5V)

\[ U_D = \frac{N_{dec}}{2^n} FS = \frac{156}{256} \times 5 = 3.0469V \]

\[ U_{in} = 3.0469 \pm 0.0098V \]
Example: 12 bit converter

Resolution = \frac{1}{2^{12}} FS = \frac{FS}{4096}

<table>
<thead>
<tr>
<th>FS</th>
<th>Resolution</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 to 10 V</td>
<td>2.44 mV</td>
</tr>
<tr>
<td>0 to 5 V</td>
<td>1.22 mV</td>
</tr>
<tr>
<td>0 to 2.5 V</td>
<td>0.61 mV</td>
</tr>
<tr>
<td>0 to 1.25 V</td>
<td>0.30 mV</td>
</tr>
<tr>
<td>0 to 1 V</td>
<td>244 μV</td>
</tr>
<tr>
<td>0 to 0.1 V</td>
<td>24.4 μV</td>
</tr>
<tr>
<td>0 mV to 20 mV</td>
<td>4.88 μV</td>
</tr>
</tbody>
</table>

Digital/Analog (D/A) Converter

\[ U_{D/A} = \frac{N_{\text{dec}}}{2^n} FS \]

Ex. FS = 5V, n = 4, code = 1111:

\[ N_{\text{dec}} = 15 \]

\[ U_{D/A} = \frac{15}{2^4} \times 5 = 4.69V \]
D/A converter: binary weighted ladder

\[ \text{code}(N_{\text{dec}}): a_1 a_2 a_3 \ldots a_{n-1} a_n \]

- Each input resistor is twice the value of the previous one
- Inputs are weighted according to their resistors

\[ V_{\text{out}} = -IR_f = -R_f \left( \frac{a_1 V_{\text{ref}}}{R} + \frac{a_2 V_{\text{ref}}}{2R} + \frac{a_3 V_{\text{ref}}}{4R} + \ldots + \frac{a_n V_{\text{ref}}}{2^{n-1} R} \right) \]

for \( R_f = R/2 \):

\[ V_{\text{out}} = -V_{\text{ref}} \left( \frac{a_1}{2} + \frac{a_2}{4} + \frac{a_3}{8} + \ldots + \frac{a_n}{2^n} \right) \]

\[ V_{\text{out}} = \frac{1}{2^n} \left( a_1 \cdot 2^n + a_2 \cdot 2^{n-1} + a_3 \cdot 2^{n-2} + \ldots + a_n \right) \]
D/A converter: binary weighted ladder

Major disadvantage:
- needs a large range of resistors with high precision (2048:1 for 12-bit DAC)
- this limits it to 4-8 bit in practice

R-2R resistor ladder

- only two resistor values (R and 2R)
- does not require high precision resistors
R-2R resistor ladder

\[ V_3 = \frac{1}{2} V_2 \]

likewise:

\[ V_2 = \frac{1}{2} V_1 \]

\[ V_1 = \frac{1}{2} V_{\text{ref}} \]

\[ V_{\text{out}} = -IR \]
R-2R resistor ladder

\[ V_3 = \frac{1}{8} V_{\text{ref}}, \quad V_2 = \frac{1}{4} V_{\text{ref}}, \quad V_1 = \frac{1}{2} V_{\text{ref}} \]

\[ V_{\text{out}} = -V_{\text{ref}} \left( \frac{a_1}{2} + \frac{a_2}{4} + \frac{a_3}{8} + \frac{a_4}{16} \right) \]

likewise:

\[ V_2 = \frac{1}{2} V_1 \]

\[ V_1 = \frac{1}{2} V_{\text{ref}} \]

\[ V_{\text{out}} = -IR \]

Successive approximation ADC

- Basic elements
  - digital to analog converter
  - analog comparator
  - control logic module
  - register

conversion time = \( n/f_0 \)
Successive approximation ADC

- Conversion time: \( n/f_0 \)
- \( V_{\text{ref}} = 10 \text{V} \)

Sample and hold (SH) circuits

- Used in the input stage of A/D converters
- Captures the voltage of a varying analog signal and keeps it at a constant level during the sampling time
Example

• We would like to convert a sinusoidal signal with the frequency $f$ using a successive approximation converter with $n$ bits and clock frequency $f_o$. Calculate a frequency above which we need to use a S/H circuit ($n = 12$, $f_o = 1$MHz)
  - Conversion time $t_c = n/f_o$
  - $u(t) = \hat{U}\cos(2\pi ft)$

**Condition**: change of $u(t)$ during $t_c \leq$ less than the quantization error
  a smaller change of signal would not change the outcome of digitization

$$\Delta u(t)_{max} \leq \frac{1}{2^n} \frac{FS}{2^n}$$

$$\Delta u(t)_{max} = \frac{du(t)}{dt} \Delta t = \frac{du(t)}{dt} t_c = 2\pi f \hat{U}_{max} t_c = 2\pi f \frac{FS}{2} t_c \leq \frac{1}{2} \frac{FS}{2^n}$$

- Answer: $t_c = 12\mu s$, $f_{limit} = 3.2$ Hz

Multiplexing

• Measurement instruments often have multiple inputs and outputs
• Instead of putting an A/D or D/A converter for every input/output, we can use multiplexing:
  - use an electronic switch for selecting input/output
  - antialiasing and reconstruction filters for each input/output
Input multiplexing

Input multiplexing with SH
Output multiplexing

Key points

- The conversion from analog to digital forms requires sampling
- Sampling frequency $f_s > 2f_{\text{max}}$
- In order to eliminate components with undesired frequencies, the signal can be filtered using a low-pass filter (antialiasing filter) with a cut-off frequency $f_c < f_s/2$
- Another low-pass filter allows us to reconstruct the signal by removing the high-frequency components due to sampling
- AD/DA converters
- SH circuits reduce conversion errors
- Multiplexing reduces the number of A/D and D/A converters and saves money
Chapter 5: Data Analysis

Measurement chain

Data analysis (recording, averaging, etc.)

Acquisition (Analog – digital conversion)

Arduino UNO board

Conditioning circuit

Sensor

Modeling

Noise reduction

Conditioning

\[ U_0 = 5 \text{ V} \]

\[ U_{out} \]

\[ R_{load} \]

\[ R_{sensor} \]
The attributes of error

- Let $x_0$ be the real value and $x$ the measured value
- Error:
  - Absolute $\Delta x = x - x_0$
  - Relative $\Delta_r x = \frac{x - x_0}{x_0}$
- Nature of the error
  - Systematic – known origin, is repeatable, can be corrected
  - Random – stochastic phenomena (noise), cannot be corrected
- Magnitude of the error
  - Maximal error – absolute limits
  - Probable error – limits associated with a given probability
Systematic error - example

- Voltage measurement:
  - $R_s$ – internal resistance of the voltage source (sensor)
  - $R_{input}$ – input resistance of the instrument (voltmeter)

$$u_m = \frac{R_{in}}{R_{in} + R_s} \cdot u_s = \frac{1\,\Omega}{100\,\Omega + 1\,\Omega} = 0.91 \times u_s$$

- Offset
  $$u_m = u + \varepsilon$$
  $\varepsilon$: systematic error

- Non-linearity
  - Known and can be corrected

Systematic error - example
Examples of random error

- Electronic noise
- Interference: EM fields, ground loops
- Temperature changes, motion of conductors
- Drift
- Hysteresis
- Repeatability

Example: temperature in the room

Error estimate

- Reporting a measurement result

\[ x = \bar{x} \pm \Delta x \ (p = p_0) \]

- \( \bar{x} \) Central tendency (usually average, mean value)
- \( \Delta x \) Incertitude
- \( p \) probability that the value \( x \) is in the range \( \bar{x} - \Delta x, \bar{x} + \Delta x \)

How to estimate the error:
- Collect samples
- Estimate the central tendency
- Estimate the dispersion interval
- Estimate the incertitude and its probability
Estimating the central tendency

• Average (arithmetic mean)

\[ \bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i \]

\[ \bar{a} = \frac{1}{N} \sum_{i=1}^{N} a_i = 9.8 \text{ m/s}^2 \]

Other estimates of the central tendency

• Geometric mean
  - Sensitive to extreme values

\[ m_g = \left( \prod_{i=1}^{N} x_i \right)^{\frac{1}{N}} \]

• Harmonic mean
  - Sensitive to extreme values

\[ m_h = \frac{N}{\sum_{i=1}^{N} \frac{1}{x_i}} \]

• Median
  - Less sensitive to extreme values

Sort \( x_i \) (measurement results) from lowest to highest

Example:
  
  Average (2,2,3,4,3,10,10) = 4.9
  
  Median (2,2,3,4,3,10,10) = Median (2,2,3,3,4,10,10) = 3
Example


Branche économique:
- 72. Recherche-développement scientifique
- Région lémanique (VD, VS, GE)
- 28. Recherche et développement
- Travaux les plus exigeants et les plus difficiles
- Cadre inférieur

Région:
- [ ]

Activité:
- [ ]

Niveau de qualification:
- [ ]

Position professionnelle:
- [ ]

Temps de travail (heures):
- [ ]

Taux d’occupation [1 - 100%]
- [ ]

Horaire hebdomadaire pour un poste à plein temps [par ex. 41,50]
- [ ]

Votre horaire hebdomadaire [par ex. 32,80]
- [ ]

Formation:
- Haute école universitaire (UNI, EPF)
- [ ]

Age:
- [ ]

Années de service:
- [ ]

Statut de séjour:
- Suisse
- [ ]

Taille de l’entreprise:
- [ ]

L’entreprise paie-t-elle 12 ou 13 salaires mensuels?
- [ ]

Touchez-vous des paiements spéciaux?
- [ ]

Etes-vous payé à l’heure ou au mois?
- [ ]

Salaires mensuels:

Salaires mensuel brut (médiane) des femmes:
- 5772 francs
- (41.50 h / semaine)

Dispersion des salaires chez les femmes:
- 4628 francs
- 7230 francs

<table>
<thead>
<tr>
<th>25%</th>
<th>50%</th>
<th>25%</th>
</tr>
</thead>
</table>

Dispersion des salaires chez les hommes:
- 6706 francs
- (41.50 h / semaine)
Distribution of measurements

- Central tendency
  - No information on the distribution
- Histogram
  - Graphical representation of the distribution of measurement results:
    - Central, peak value
    - Range of values
    - Dispersion
    - Presence of extreme values
    - Distribution profile

Histogram - example

- 400 samples with a resistance of 100 Ω are measured using an ohm-meter
  - The range of results is divided in \( k \) classes (bins): \( x \text{ axis} \)
  - Number of results in each class (bin): \( y \text{ axis} \)
  - Number of samples: \( N \)
  - Good choice for the number of classes: \( k = \sqrt{N} \)
Examples of histograms

Symmetric: walking speed of young people

Nonsymmetric: duration of walking periods during the day

Symmetric: walking speed of young people and seniors

Estimate of dispersion

- Variance
  \[ \text{Var}(x) = \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2 \]

- Standard deviation
  \[ \sigma = \sqrt{\text{Var}(x)} = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2} \]

- For \( N < 30 \) or a limited number of measurements
  \[ \sigma = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})^2} \]

- Other estimates of dispersion:
  - \( z \) score
    \[ z = \frac{x - \bar{x}}{\sigma} \]
  - Range: \( x_{\text{max}} - x_{\text{min}} \)
Model for dispersion

- Normal distribution (Gaussian distribution)

\[ f(x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{x-\bar{x}}{\sigma_x}\right)^2} \]

Measuring dispersion

\[ \pm \sigma_x = \pm 0.9 \text{m/s}^2 \]

\[ \bar{x} = 9.8 \text{m/s}^2 \]

\[ \sigma_x^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2 = 0.81 \text{m}^2 / \text{s}^4 \]

\[ \sigma_x = 0.9 \text{m/s}^2 \]
Error Probability

- Frequency
  \[ f(x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma_x}\right)^2} \]

- Cumulative frequency
  \[ F(x) = \int_{-\infty}^{x} f(x')dx' \]

- Probability
  \[ P(-\infty < x < \infty) = \int_{-\infty}^{+\infty} f(x)dx = 1 = 100\% \]
  \[ P(x \leq a) = \int_{-\infty}^{a} f(x)dx = F(a) \]
  \[ P(b \leq x \leq a) = \int_{b}^{a} f(x)dx = F(a) - F(b) \]
Standard normal distribution

\[ f(x) = p(x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x-\mu}{\sigma_x} \right)^2} \]

\[ f(z) = p(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2} \]

\[ z = \frac{x - \bar{x}}{\sigma}, \sigma_z = 1 \]

\[ P(-z) = 1 - P(z) \]

**Example** \( P(-1 < z < 1) = 0.84 - 0.16 = 0.68 \)

\[ P(1) = 0.84 \]
\[ P(-1) = 0.16 \]

Interactive Example

11.1.1 Table de la distribution normale standard $N(0,1)$

$$p(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$$

<table>
<thead>
<tr>
<th>$z$</th>
<th>$\alpha$</th>
<th>$z$</th>
<th>$\alpha$</th>
<th>$z$</th>
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</table>

Error and its probability

$\sigma_x: 0.11 m/s^2$

$\pm 2\sigma$: 95% of samples

$\pm 3\sigma$: 99.7% of samples

Incertitude = $\sigma$ with a confidence level of 68 %

Incertitude = $2\sigma$ with a confidence level of 95 %

Incertitude = $3\sigma$ with a confidence level of 99 %
### Precision and accuracy

- **Precision** (also reproducibility, repeatability) – degree to which repeated measurements under unchanged conditions show the same results

- **Accuracy** – closeness of the measurement result to the true value

---

**Neither precise, nor accurate**  
**True value**  
**measured value**

---

**Neither accurate, nor precise**  
**F(x)**

---

**Accurate, not precise**  
**F(x)**

---

**Not accurate, precise**  
**F(x)**
Estimation of systematic and random error

- **Systematic error:**
  - estimate the central tendency: average, median
  - compare to a reference system
  - hypothesis test (more on this in chapter 6)

- **Random error:**
  - Estimation of the dispersion: standard deviation
  - Estimation of the probability of the error

Example: temperature measurement in a room

\[ \bar{x} = T_{\text{average}} = 27^\circ C \]

\[ \sigma_x = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2} = 0.9 \, ^\circ C \]

\[ T \text{ in the room } \mu = 25^\circ C \] (known)

**Systematic error:** 2\(^\circ\)C

**Random error:** \( \pm \sigma_x = \pm 0.9^\circ C \) (p=0.68)
\( \pm 2\sigma_x = \pm 1.8^\circ C \) (p=0.95)
\( \pm 3\sigma_x = \pm 2.7^\circ C \) (p=0.99)
Example

Let us consider Johnson noise from a 20 MΩ resistor at room temperature measured in a bandwidth of 10 MHz. What is the probability of measuring a voltage above 1mV at any given time at room temperature?

\[ \overline{u_n^2} = 4k_B TR \]  
Power spectral density of voltage variance

\[ \overline{U_n^2} = 4k_B TR \cdot B = 3.23 \times 10^{-5} \, V^2 \]

\[ \sqrt{U_n^2} = 1.79 \, mV = \sigma \]

\[ z = \frac{x - \mu}{\sigma} = \frac{1mV - 0}{1.79mV} = 0.56 \]

\[ z = 0.56 \]

\[ \sim 29\% \]

Increasing the precision

Example: measurement of gravity using an accelerometer

First approach:
- Acquire 500 points
- Calculate average and s.dev.:

**Individual measurement:**
\[ a = \overline{a} \pm \sigma_a \quad (p = 0.68) \]

**Error for an individual measurement**
\[ \sigma_a = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (a_i - \overline{a})^2} = 0.1m / s^2 \]

\[ a = 9.81 \, m / s^2 \pm 0.1m / s^2 \quad (p = 0.68) \]
Increasing the precision

Example: measurement of gravity using an accelerometer

Second approach:
- Take those same 500 points
- During acquisition or analysis, bunch the points in groups of for example 40 points
- Calculate the averages for every group separately -> $\bar{a}_{N=40}$
- Calculate the average of group averages \[ \bar{a}_N = \frac{1}{N} \sum_{k=1}^{N} \bar{x}_k = \bar{x} \]

\[ \sigma_{\bar{a}_N} = \frac{\sigma}{\sqrt{N}} = \frac{0.1}{\sqrt{40}} = 0.016 \text{ m/s}^2 \]

\[ \bar{a}_N = \bar{a}_N \pm \sigma_{\bar{a}_N} = \bar{a} \pm \frac{\sigma_a}{\sqrt{N}} \]

Error of the «local» average

«Local» average

Estimation of the local average

• Calculating the “local” average allows us to reduce the random error and increase the precision of the measurement

• Compare $Var(x)$ and $Var(\bar{x})$

• $N$ – number of points in a subset, $M$ - number of subsets, $N \times M$ – total number of points

\[ Var(\bar{x}) = \frac{1}{M} \sum_{j=1}^{M} \left( \frac{1}{N} \sum_{i=1}^{N} x_{ji} - \bar{x} \right)^2 = \frac{1}{M} \frac{1}{N^2} \sum_{j=1}^{M} \left( \sum_{i} x_{ji} - N\bar{x} \right)^2 = \frac{1}{M} \frac{1}{N^2} \sum_{j=1}^{M} \left( \sum_{i} x_{ji} - \bar{x} \right)^2 \]

\[ = \frac{1}{M} \frac{1}{N^2} \sum_{j} \left[ \sum_{i} (x_{ji} - \bar{x})^2 + \sum_{i} (x_{ji} - \bar{x}) (x_{ji} - \bar{x}) \right] = \frac{1}{M} \frac{1}{N^2} \sum_{j} \left[ \sum_{i} (x_{ji} - \bar{x})^2 \right] = \frac{1}{M} \frac{1}{N^2} \sum_{j} \left[ \sum_{i} (x_{ji} - \bar{x})^2 \right] \]

\[ = \frac{1}{M} \frac{1}{N^2} \sum_{k} (x - \bar{x})^2 = \frac{1}{M} \frac{1}{N^2} M \cdot N \cdot Var(x) = \frac{Var(x)}{N} \]
Increasing $N \rightarrow$ increasing precision (decreasing dispersion)

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}$$

Question

In practice, you would not take 1000 points but you would have a sensor that gives a single reading with an error $\sigma$ with probability of 68% (this is usually specified by the manufacturer).

How many readings $N$ do we need in order to estimate the average value with error $\delta$ and associated probability $p$?
For a normal distribution 68% of the points are between ± σ

- **standard deviation of the average** < standard deviation of the population (single measurement)
- Error of the estimation = standard deviation of the average
- **p = 1−2α** – probability that the difference between a single reading and the average of the readings is less than δ

**Confidence interval of the average**

1. **σ_\bar{x}**
   - confidence level of 68%
   - $CI_{68\%} = \left[ \bar{x} - \frac{\sigma}{\sqrt{N}}, \bar{x} + \frac{\sigma}{\sqrt{N}} \right]$

2. **2 \cdot σ_\bar{x}**
   - confidence level of 95%
   - $CI_{95\%} = \left[ \bar{x} - 2 \cdot \frac{\sigma}{\sqrt{N}}, \bar{x} + 2 \cdot \frac{\sigma}{\sqrt{N}} \right]$

3. **z \cdot σ_\bar{x}**
   - confidence level of p(z) (see in the table)
   - $CI_{p\%} = \left[ \bar{x} - z_\alpha \cdot \frac{\sigma}{\sqrt{N}}, \bar{x} + z_\alpha \cdot \frac{\sigma}{\sqrt{N}} \right]$

$z = \frac{\bar{x} - \mu}{\sigma_\bar{x}}$

$\alpha = (1-p)/2$
Example: error of a gyroscope

- In order to estimate the error of a gyroscope reading, we carry out 36 identical tests. They consist of turning the gyroscope by 360°, taking the angular velocity readings while the gyroscope is turning and then integrating the velocity (which should give us a total angle of 360°)
- We get $\bar{x} = 359.8\, \text{deg}; \sigma = 2.4\, \text{deg}$
- Calculate the 95% confidence interval for this test (the interval in which the value of the next test with 36 readings would be, with a 95% probability)
There is a probability of 95% of getting the next average between 359.0 and 360.6 deg (358.8 ± 0.8)
Normal distribution of the average

• If the standard deviation of an individual measurement is known and if the total number of measurements $N > 30$, we use the normal distribution

  • Confidence interval: $\bar{x} - z_{\alpha} \frac{\sigma}{\sqrt{N}} < \mu < \bar{x} + z_{\alpha} \frac{\sigma}{\sqrt{N}}$

  ![Diagram of confidence interval for normal distribution]

  $\alpha = (1-p)/2$

  $p$: probability

  with $p(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$

Student distribution of the average

• Proposed by William Sealy Gosset in 1908

• If the standard deviation of an individual measurement is estimated or if the total number of measurements $N < 30$, we use the Student distribution

  • Confidence interval

  $\bar{x} - t_{\alpha} \frac{\sigma}{\sqrt{N}} < \mu < \bar{x} + t_{\alpha} \frac{\sigma}{\sqrt{N}}$
Student distribution of the average

\[ p(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\nu\pi}\left(1+\frac{t^2}{\nu}\right)^{\nu+1/2}} \]

[For normal distribution, we had \( p(z) \)]

\[ \Gamma(y) = \int_0^{+\infty} x^{y-1}e^{-x} \, dx \]

\( \nu \): number of degrees of freedom = \( N - 1 \) where \( N \) is the number of measurements

### Table de la distribution \( T \) de Student

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<th>( \nu )</th>
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<th>0,25</th>
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<th>0,05</th>
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Sample size – normal distribution

• How many points should we acquire for estimating the average with an error $\delta$ and probability $p$?

• Case 1: we know $\sigma$ (the sdev. of a single measurement). We therefore use the normal distribution

$$\alpha = \frac{1-p}{2} \quad \bar{x} = \mu \pm \delta = \mu \pm z_\alpha \frac{\sigma}{\sqrt{N}}$$

1. Define the error: $\delta = \bar{x} - \mu = z_\alpha \frac{\sigma}{\sqrt{N}}$

2. Find $z$ from $\alpha$ (table)

3. Number of points $N$ is then:

$$N = z_\alpha^2 \left( \frac{\sigma}{\delta} \right)^2$$

Example

• Two sensors have noisy outputs with $U_1 = 9.8 \text{ V} \ (\sigma_1 = 1 \text{ V})$ and $U_2 = 9.35 \text{ V} \ (\sigma_2 = 1.32 \text{ V})$. Calculate the smallest number of samples so that values of $U_1$ and $U_2$ are within 1% of the actual value with a confidence of 90% and 99%?

$$N = z_\alpha^2 \left( \frac{\sigma}{\delta} \right)^2 \quad z_{5\%} = 1.65; \ z_{0.5\%} = 2.58; \ \delta_1 = 0.098 \text{ V}, \ \delta_2 = 0.0935 \text{ V}$$

90%:

$$N_1 = 1.65^2 \left( \frac{1}{0.098^2} \right) = 283 \quad N_2 = 1.65^2 \left( \frac{1.32^2}{0.0935^2} \right) = 543$$

99%:

$$N_1 = 2.58^2 \left( \frac{1}{0.098^2} \right) = 693 \quad N_2 = 2.58^2 \left( \frac{1.32^2}{0.0935^2} \right) = 1327$$
Error estimate - repeated

• Reporting a measurement result

\[ x = \bar{x} \pm \Delta x \quad (p = p_0) \]

- \( \bar{x} \) Central tendency (usually average, mean value)
- \( \Delta x \) Incertitude
- \( p \) probability that the value \( x \) is in the range \( \bar{x} - \Delta x, \bar{x} + \Delta x \)

How to estimate the error:
- Collect samples
- Estimate the central tendency
- Estimate the dispersion interval
- Estimate the incertitude and its probability

Estimate of dispersion - repeated

• Variance

\[ Var(x) = \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2 \]

• Standard deviation

\[ \sigma = \sqrt{Var(x)} = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2} \]

• For \( N<30 \) or a limited number of measurements

\[ \sigma = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})^2} \]

• Other estimates of dispersion:

- \( z \) score \( z = \frac{x - \bar{x}}{\sigma} \)
- Range: \( x_{max} - x_{min} \)
Model for dispersion - repeated

- Normal distribution (Gaussian distribution)

\[ f(x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x - \bar{x}}{\sigma_x} \right)^2} \]

Error and its probability - repeated

\[ \text{Incertitude} = \sigma \text{ with a confidence level of 68%} \]
\[ \text{Incertitude} = 2 \sigma \text{ with a confidence level of 95%} \]
\[ \text{Incertitude} = 3 \sigma \text{ with a confidence level of 99%} \]
Normal distribution of the average - repeated

• If the standard deviation of an individual measurement is known and if the total number of measurements $N > 30$, we use the normal distribution

• Confidence interval: $\bar{x} - z_\alpha \frac{\sigma}{\sqrt{N}} < \mu < \bar{x} + z_\alpha \frac{\sigma}{\sqrt{N}}$

$$\alpha = (1-p)/2$$
$p$: probability

Sample size – normal distribution - repeated

• How many points should we acquire for estimating the average with an error $\delta$ and probability $p$?

• Case 1: we know $\sigma$ (the sdev. of a single measurement). We therefore use the normal distribution

$$\alpha = \frac{1-p}{2} \quad \bar{x} = \mu \pm \delta = \mu \pm z_\alpha \frac{\sigma}{\sqrt{N}}$$

1. Define the error: $\delta = \bar{x} - \mu = z_\alpha \frac{\sigma}{\sqrt{N}}$
2. Find $z$ from $\alpha$ (table)
3. Number of points $N$ is then:

$$N = z_\alpha^2 \left( \frac{\sigma}{\delta} \right)^2$$
Sample size – Student distribution

• How many points should we acquire for estimating the average with an error \( \delta \) with probability \( p \)?
• Case 2: we do not know \( \sigma \) (the s. dev. of a single measurement or in other words we do not know the specs of the sensor)

\[
\alpha = \frac{1 - p}{2}
\]

1. Choose an initial sample size \( N' \) by performing \( N' \) measurements
2. Estimate \( \bar{x} \) and \( \sigma \)
3. Find \( t_\alpha \) from the table with \( v = N' - 1 \)
4. Optional: calculate error \( \delta' \) (the error for this set of measurements):
   \[
   \delta' = t_\alpha \frac{\sigma}{\sqrt{N}}
   \]
5. Calculate desired sample size \( N \):
   \[
   N = t_\alpha^2 \left( \frac{\sigma}{\delta} \right)^2
   \]

Example

• Based on 6 measurements, the average melting temperature of tin is estimated to be 232.26 °C with a standard deviation of 0.14°C. If we use this value as the real melting temperature, calculate the maximum error with a confidence level of 98%. How many measurements should we make to have an error of 0.05 °C?

\[
N = 6, \ \sigma = 0.14, \ t_{0.01} = 3.365 \ (for \ v = N - 1 = 5)
\]

\[
\delta = t_{0.01} \left( \frac{\sigma}{\sqrt{N}} \right) = 3.365 \left( \frac{0.14}{\sqrt{6}} \right) = 0.19°C \quad \text{Error} = 0.19°C (p=98%)
\]

\[
N = 3.365^2 \left( \frac{0.14}{0.05} \right)^2 = 89
\]
### Precision of digital instruments

\[ u_a = a \cdot u_m + b \]

\[ u_a + \Delta u_a = (a + \Delta a)u_m + b + \Delta b \]

\[ \Delta u_a = \Delta a \cdot u_m + \Delta b \]

**Error:** \( \pm \text{(% reading + n digit)} \)

**Error on the gain:**

**Error on the offset:**

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<td>2.069</td>
<td>2.500</td>
<td>2.807</td>
<td>3.104</td>
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<td>3.767</td>
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<tr>
<td>24</td>
<td>0.256</td>
<td>0.685</td>
<td>1.318</td>
<td>1.711</td>
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<td>2.492</td>
<td>2.797</td>
<td>3.091</td>
<td>3.467</td>
<td>3.745</td>
</tr>
</tbody>
</table>
Precision of digital instruments

- Error: ± (% reading + n digit) -
  - for example BBC M2030: ± (0.1%reading+1d)

For displayed value of 4.00 V:
  1d : 0.01 V
  0.1%reading = 0.1% × 4 V = 4 mV
  Error = ±(4 mV + 0.01 V) = ±0.014 V
  Relative error = ±0.014/4 = 0.35%

- Error: ± (%reading + %FS)
  - for example Philips 2514 voltmeter: ± (0.1%reading + 0.02%FS)

For displayed value of 4.000 V:
  0.02%FS (= 10 V) = 0.002 V
  Error: ±(0.1% × 4 V + 0.002 V) = ±0.006 V

  Relative error: ±0.006 V / 4 V = 0.15%
Key points

• Determining the precision of a measurement and its probability from the systematic and random errors
• Estimation of the systematic error from the central tendency and the random error from the dispersion
• Using normal and Student distributions for modelling the dispersion
• Calculating the error on the average
• Estimating the confidence interval for an average
• Estimating the sample size
Chapter 6: Comparison

Measurement chain

- Acquisition (Analog – digital conversion)
- Conditioning
- Data analysis (recording, averaging, etc.)
- Noise reduction
- Sensor
- Modeling
Chapter 6: Comparing measurement results

- Dispersion diagram
- Regression and correlation
- Hypothesis testing
Examples of questions to answer

• Metrology
  Q1: does the average value supplied by the sensor correspond to the actual (real) value we are trying to measure?
  Q2: which one of two or more measurement methods is more precise (smaller $\sigma$) / correct (closer to the real value)?
  Q3: which of the two or more noise reduction methods is more efficient (results in a smaller $\sigma$)?

• Other domains
  - effectiveness of a medical treatment
  - differences between populations

Dispersion diagram

• Presentation of ($x,y$) coordinate pairs
• Highlighting a relationship
• Statistical distribution?
• Total number of measurements?
Regression and correlation

• Identifying a linear relationship between $x$ and $y$
• Linear regression line $y_e = ax + b$

Minimize $D = \sum_{i=1}^{N} (y_i - y_e)^2$
\[ \frac{\partial D}{\partial a} = 0, \quad \frac{\partial D}{\partial b} = 0 \]

\[ a = \frac{\sum_{i=1}^{N} x_i y_i - \frac{1}{N} \sum_{i=1}^{N} x_i \sum_{i=1}^{N} y_i}{\sum_{i=1}^{N} x_i^2 - \frac{1}{N} \left( \sum_{i=1}^{N} x_i \right)^2} \]

\[ b = \bar{y} - a\bar{x} = \frac{\frac{1}{N} \sum_{i=1}^{N} x_i^2 \sum_{i=1}^{N} y_i - \frac{1}{N} \sum_{i=1}^{N} x_i \sum_{i=1}^{N} x_i y_i}{\sum_{i=1}^{N} x_i^2 - \frac{1}{N} \left( \sum_{i=1}^{N} x_i \right)^2} \]

Regression and correlation

• All these data sets result in the same regression line:

Linear relationship  Dependence  No dependence

• How do we measure the significance of the regression line, how faithfully it represents the original data?
Correlation coefficient

- $R$ measures the strength of the linear relationship

$$R = \frac{s_{xy}^2}{s_x s_y} = \frac{S_x}{S_y}$$

$$s_{xy}^2 = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})$$

$$-1 < R < 1$$

$s_x^2$: estimates the variance of $x$ ($\sigma_x^2$)
$s_y^2$: estimates the variance of $y$ ($\sigma_y^2$)
$s_{xy}^2$: estimates the covariance of $x$ and $y$

- $R$ does not measure the strength of a non-linear relationship!

Examples of correlation

$N=11$, $R=0.81$

Source: wikipedia
Correlation and causation

Correlation does not mean causation: correlation is necessary but not sufficient.

Source: Gizmodo
Correct use of regression

- Show the 95% confidence interval
  - Calculate the standard deviation of the difference $s_e$
  - Trace the zones of $\pm 1.96 s_e$ (normal distribution, $z = 1.96$ for $\alpha = 2.5\%$)

$$s_e = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (y_i - y_{ei})^2}$$

Hypothesis test

Q1: Does the average value supplied by the sensor correspond to the actual value we are trying to measure?

How do we answer this?

By doing a hypothesis test which consists of:
- Making the initial assumption
- Collecting evidence (data)
- Based on the available evidence (data), deciding whether to reject or not reject the initial assumption.
Hypothesis test

Q1: Does the average value supplied by the sensor correspond to the actual value?

• We know the theoretical average (actual value, $\mu$) of a population and its confidence interval:

$$\mu - z_\alpha \frac{\sigma}{\sqrt{N}} < \mu < \mu + z_\alpha \frac{\sigma}{\sqrt{N}}$$

• We collect $N$ samples from this population (make $N$ measurements), calculate the average and find:

$$\bar{x} \neq \mu$$

• The question now is whether the average $\bar{x}$ is significantly different from the actual value $\mu$ or if the difference is due to chance.

Hypothesis test

• $\bar{x}$ is significantly different from the actual value $\mu$ -
  - $\bar{x}$ is outside the confidence interval

• The difference between $\bar{x}$ and $\mu$ is due to chance
  - $\bar{x}$ is inside the confidence interval

![Diagram showing the relationship between the confidence interval, the actual value $\mu$, and the sample mean $\bar{x}$, with $\alpha$ marking the significance level.](image)
Hypothesis test - example

- Specification sheet: the voltage source provides $\mu = 10$ V with a $\sigma = 0.2$ V (This is the claim, hypothesis)
- How do we test this?
  
  - Perform $N = 100$ measurements ($N$ can be any large number of measurements $> 30$ so we can apply the normal distribution) and calculate $\bar{x}$
  
  - Estimate if $\bar{x}$ is in the specified confidence interval

  - If $\bar{x} > \mu + z_\alpha \frac{\sigma}{\sqrt{N}}$ or $\bar{x} < \mu - z_\alpha \frac{\sigma}{\sqrt{N}}$
    - systematic error

  - If $\mu - z_\alpha \frac{\sigma}{\sqrt{N}} \leq \bar{x} \leq \mu + z_\alpha \frac{\sigma}{\sqrt{N}}$
    - difference due to chance

- For $p = 95\%$ $z_{\alpha=0.025} = 1.96$

- For $\bar{x} = 10.1$ V
  
  \[ z_{obs} = \frac{\bar{x} - \mu}{\sigma/\sqrt{N}} = \frac{10.1 - 10}{0.2/10} = 5 > z_\alpha \]

  - We reject the hypothesis (claim) of the manufacturer. We are 95% sure that the difference between $\bar{x} = 10.1$ V and $\mu = 10$ V is significant.

- For $\bar{x} = 10.03$ V
  
  \[ z'_{obs} = \frac{\bar{x} - \mu}{\sigma/\sqrt{N}} = \frac{10.03 - 10}{0.2/10} = 1.5 < z_\alpha \]

  - We do not reject the hypothesis of the manufacturer. We can however not claim that the source actually delivers 10V, only that the difference is due to chance (random error).

- A hypothesis is never accepted: only rejected or not rejected.
Usefulness of the hypothesis test

• How do we know if the hypothesis on the measured values is right or probable
  • Example: Hypothesis – the value provided by the voltage source is 10.01 V
  • How do we check this?
    - do an infinitely large number of measurements and calculate the average
    - perform sampling and calculate the average – SIGNIFICANCE
• Significant difference: the difference between two values is not due to chance (systematic error)
• No significant difference: the difference is due to chance (random error)

Main uses of hypothesis testing

• Comparison of an experimental average with a theoretical one
• Comparison of two experimental averages
• Comparison of two variances (precisions)
• Comparison of an experimental variance with a theoretical variance
Definition of the hypothesis test

- Data analysis procedure with the outcome of **rejecting** or **failing to reject** (not the same as accepting!) a hypothesis based on the data
- There are always two hypotheses:
  - $H_0$ – the result of an estimation does not significantly differ from the actual value (theoretical or supposed). This is the **null hypothesis**
    \[
    H_0: \quad \bar{x} = \mu \quad \text{there is no difference between the estimated average and the theoretical value}
    \]
  - $H_a$ – the result of an estimation significantly differs from the actual value (theoretical or supposed). **This is the alternative hypothesis.**
    \[
    H_a: \quad \bar{x} \neq \mu
    \]
- During a hypothesis test, we always assume $H_0$ is true and announce it in the form of a sentence: Example – Can we say that the voltage source provides 10V?
  - $H_0$ – “The estimated average is not different from 10V”
  - $H_0$ – “The difference between the estimated value and 10V is zero.”
  - $H_0$ – “The voltage source provides 10V.”

Bilateral test

- Used when we do not know in advance the particular direction of the alternative hypothesis (if $\bar{x} > \mu$ or $\bar{x} < \mu$)
- Use $\alpha$ to determine the risk of error which is $\beta = 2\alpha$
Flowchart for the realisation of a bilateral test

- Formulate the hypothesis $H_0$
- Define the statistical distribution ($z$)
- Set the risk $\beta=2\alpha$
- Calculate $z_\alpha$
- Calculate $z_{obs}$
- Compare with $CI_{(1-2\alpha)}$
- If $|z_{obs}|>|z_\alpha|$ - reject $H_0$
- Formulate the conclusion

A sentence expressing the lack of difference:
“The voltage source is providing 10V”
or “The tension of the voltage source is not different from 10V”

in general 1%, 5%, 10%

tables

$z_{obs} = \frac{\bar{x} - \mu}{\sigma/\sqrt{N}}$

Example: bilateral z-test

- According to the specifications, a sensor should draw 2.80 mA of current with a standard deviation of 0.14 mA. To test this, we take 40 sensors and find an average current draw of 2.72 mA. What can we conclude with a risk of 5% about the specifications?

$H_0$: $\mu=2.80$ mA – The sensor draws 2.80 mA

$Z_{obs} = \frac{\bar{x} - \mu}{\sigma/\sqrt{N}}$

$\alpha = 5%/2 = 0.025$

$z_\alpha = -1.96$

$z_{obs} = (2.72-2.80)/(0.14/6.3)=-3.61$

-3.61<-1.96 : we reject $H_0$

Conclusion: the average current draw is different from 2.8 mA, with a risk of 5%.
Unilateral test

- Used when we expect the average to be above or below the theoretical average (specifications)
- Use $\alpha$ to determine the risk of error (in this case it’s $\alpha$)
- Formulate the null hypothesis

$H_0: \bar{x} > \mu$ therefore $H_a: \bar{x} < \mu$

$H_0: \bar{x} < \mu$ therefore $H_a: \bar{x} > \mu$

$z_{obs} = \frac{\bar{x} - \mu}{\sigma/\sqrt{N}}$

Flowchart for the realisation of a unilateral test

**Formulate the hypothesis $H_0 \bar{x} \geq \mu$**

- Define the statistical distribution ($z$)
- Set the risk $\alpha$
- Calculate $z_\alpha$
- Calculate $z_{obs} = \frac{\bar{x} - \mu}{\sigma/\sqrt{N}}$
- Compare with CI$(1-\alpha)$%
  - If $z_{obs} < -|z_\alpha|$ - reject $H_0$
  - Formulate the conclusion

**Formulate the hypothesis $H_0 \bar{x} \leq \mu$**

- Define the statistical distribution ($z$)
- Set the risk $\alpha$
- Calculate $z_\alpha$
- Calculate $z_{obs} = \frac{\bar{x} - \mu}{\sigma/\sqrt{N}}$
- Compare with CI$(1-\alpha)$%
  - If $z_{obs} > |z_\alpha|$ - reject $H_0$
  - Formulate the conclusion

6.26

6.27
Example: unilateral z-test

• According to the specifications, a sensor should draw 2.80 mA of current with a standard deviation of 0.14 mA. To test this, we take 40 sensors and find an average current draw of 2.72 mA. What can we conclude about the specifications with a 5% risk?

• Express the null hypothesis

\[ H_0 : \text{the current draw is higher than 2.80 mA} \]

(because the only thing we can do is reject a hypothesis, we can’t accept it)

6.28

\[
\begin{align*}
\bar{x} &= \frac{x_1 + x_2 + \cdots + x_N}{N} \\
\sigma &= \sqrt{\frac{\sum (x_i - \bar{x})^2}{N}} \\
\alpha &= 5\% = 0.05 \\
z_\alpha &= -1.645 \\
z_{\text{obs}} &= (2.72 - 2.80)/(0.14/\sqrt{40}) = -3.61 \\
-3.61 &< -1.645 : \text{we reject } H_0
\end{align*}
\]

Conclusion: the average current draw is smaller than 2.8 mA, with a risk of 5%.
Comparison of an experimental average with a theoretical one – t-test

- In cases where the number of measurements $N < 30$ or if the standard deviation is estimated from an experiment (and not specifications), we use the Student distribution instead of the normal one and the t-test instead of the z-test.

- The procedure is the same as in the z-test:
  - we replace $z_\alpha$ with $t_\alpha$, defined from the Student distribution
  - $t_{obs}$ is given by:
    \[ t_{obs} = \frac{\bar{x} - \mu}{s / \sqrt{N}} \]

Example: t-test

- In order to estimate the error of a gyroscope reading, we carry out 22 identical tests. They consist of turning the gyroscope by 360°, taking the angular velocity readings while the gyroscope is turning and then integrating the velocity (which should give us the total angle or 360°). We find in this way an average value of 359.2° and a standard deviation of 4.4°. Can we say with a 5% risk that the sensor is producing a systematic error?

  $H_0$: The sensor is not making a systematic error
  The difference between the result and the theoretical value is not significant

  \[ t = \frac{\bar{x} - \mu}{s / \sqrt{N}} \]

  \[ 2\alpha = 5\%; \alpha = 0.025 \text{ (bilateral test)} \]
  \[ t_\alpha = 2.08 \]
  \[ t_{obs} = \frac{(359.2 - 360)}{(4.4/4.7)} = -0.85 \]
  \[-0.85 > -2.08 : \text{we do not reject } H_0 \]

**Conclusion: the difference is not significant with a risk of 5%.**
Comparison of two experimental averages: z-test

- Comparing $\bar{x}_1$ with $\bar{x}_2$ is the equivalent of comparing $\bar{x}_1 - \bar{x}_2$ with 0.
- If $x_1$ and $x_2$ are independent, then:

$$Var(\bar{x}_1 - \bar{x}_2) = Var(\bar{x}_1) + Var(\bar{x}_2) = \frac{\sigma_1^2}{N_1} + \frac{\sigma_2^2}{N_2}$$

- The variable $z$ follows the normal distribution

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{N_1} + \frac{\sigma_2^2}{N_2}}}$$

- If the standard deviations have been determined from the experiment or $N_1 < 30$ or $N_2 < 30$, we use the variable $t$ and the Student distribution

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{s_1^2/N_1 + s_2^2/N_2}}$$

Example

- Two sensors have noisy outputs with $U_1 = 9.8$ V ($\sigma_1 = 1$ V) and $U_2 = 9.6$ V ($\sigma_2 = 1.32$ V). Can we say with a risk of 2% that the values $U_1$ and $U_2$ are different after taking 500 measurements?

$H_0$: The two averages are not different

$$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{N_1} + \frac{\sigma_2^2}{N_2}}}$$

$\alpha = 2%/2 = 0.01$ (bilateral test)

$z_{1%} = -2.33$

$z_{obs} = (9.8-9.6)/(1.66/22.4)=2.69$

2.69>2.33 : we reject $H_0$

Conclusion: the averages are significantly different, with a risk of 2%.
Comparison between two variances

- We can also compare two sets of data according to their variance

<table>
<thead>
<tr>
<th></th>
<th>Same average</th>
<th>Different average</th>
</tr>
</thead>
<tbody>
<tr>
<td>Same variance</td>
<td><img src="image1.png" alt="Graph" /></td>
<td><img src="image2.png" alt="Graph" /></td>
</tr>
<tr>
<td>F-test</td>
<td><img src="image3.png" alt="Graph" /></td>
<td><img src="image4.png" alt="Graph" /></td>
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<tr>
<td>Different variance</td>
<td><img src="image5.png" alt="Graph" /></td>
<td><img src="image6.png" alt="Graph" /></td>
</tr>
</tbody>
</table>

Comparing two measured variances

- Let us assume that we have two sets of data (populations) with distributions that can be described using the normal (Gaussian) distribution
- Let $s_1^2$ and $s_2^2$ be the variances estimated using $N_1$ and $N_2$ samples with $s_1^2 > s_2^2$
- In this case the quantity $f = \frac{s_1^2}{s_2^2}$ follows the Fisher distribution

\[
f(v_1, v_2) = \frac{\Gamma\left(\frac{v_1 + v_2}{2}\right) v_1^{v_1/2} v_2^{v_2/2} f^{(v_1/2)-1}}{\Gamma\left(\frac{v_1}{2}\right) \Gamma\left(\frac{v_2}{2}\right) (v_2 + v_1 f)^{\frac{v_1+v_2}{2}}}\]

\[
p(f) = \frac{1}{f} f_{1-\alpha}(v_1, v_2) = 1/f_{\alpha}(v_2, v_1) \]

$v_1 = N_1 - 1$
$v_2 = N_2 - 1$
Example: bilateral F-test

- Two sensors have noisy outputs with $U_1 = 9.35\, \text{V} (s_1 = 1.5\, \text{V})$ and $U_2 = 9.8\, \text{V} (s_2 = 1\, \text{V})$. Can we say with a risk of 5% that the two sensors have different noise levels? $s_1$ and $s_2$ have been calculated based on 31 measurements.

$H_0$: The two noise levels are not different

\[ f = \frac{s_2^2}{s_1^2} \]

$\alpha = 5\%/2 = 0.025$ (bilateral test)

\[ f_{1-\alpha/2}(v_1, v_2) = 1/f_{\alpha/2}(v_2, v_1) \]

\[ f_{2.5\%}(30, 30) = 2.07, \quad f_{97.5\%}(30, 30) = 0.483 \]

\[ f_{\text{obs}} = (1.5)^2 / 1 = 2.25 > 2.07: \text{we reject } H_0 \]

Conclusion: the noise levels are significantly different, with a risk of 5%.
Comparing two measured variances – F-test

### Bilateral

- **Formulate the hypothesis** \( H_0 : s_1 = s_2 \)
- **Define the test variable** \( f \)
- **Set the risk** \( 2\alpha \)
- **Calculate** \( f_\alpha \) and \( f_{1-\alpha} \)
- **Calculate** \( f_{\text{obs}} = \frac{s_1^2}{s_2^2} \)
- **Compare with CI\((1-2\alpha)\%\)**
  - If \( f_{\text{obs}} > f_\alpha \) or \( f_{\text{obs}} < f_{1-\alpha} \) – reject \( H_0 \)
- **Formulate the conclusion**

### Unilateral

- **Formulate the hypothesis** \( H_0 : s_1 \geq s_2 \)
- **Define the test variable** \( f \)
- **Set the risk** \( \alpha \)
- **Calculate** \( f_{1-\alpha} \)
- **Calculate** \( f_{\text{obs}} = \frac{s_1^2}{s_2^2} \)
- **Compare with CI\((1-\alpha)\%\)**
  - If \( f_{\text{obs}} < f_{1-\alpha} \) – reject \( H_0 \)
- **Formulate the conclusion**
Comparison between an experimental and a theoretical variance: $\chi^2$ test

- Let $\sigma^2$ be the theoretical variance and $s^2$ the experimentally determined variance, estimated using $N$ samples.
- The variable $\chi^2$ is defined as $\chi^2 = \frac{(N-1)s^2}{\sigma^2}$ with a distribution $\chi^2(\nu)$.

$$p(\chi^2) = \left[2^{\nu/2} \Gamma\left(\frac{\nu}{2}\right)\right]^{-1} (\chi^2)^{(\nu/2)-1} e^{-\chi^2/2}$$

Example: $\chi^2$ test

- An amplifier is characterised by noise $\sigma = 2.2$ $\mu$V. A filter is used at the output in order to reduce this noise. The noise amplitude after filtering is estimated to be $s = 1.92$ $\mu$V based on 31 measurements. Determine with a risk of 5% if the filter is effective in reducing the noise.

$H_0$: The noise level after filtering is higher than before the filtering $\chi^2 = \frac{(N-1)s^2}{\sigma^2}$

$\alpha = 5\%$ (unilateral test)

$\chi^2_{95\%}(30)=18.49$

$\chi^2_{\text{obs}}=(30)\times1.92^2/2.2^2 = 22.85$

$22.85 > 18.49$ : we do not reject $H_0$

Conclusion: the filter is not effective.
Comparison between an experimental and a theoretical variance: bilateral $\chi^2$-test

<table>
<thead>
<tr>
<th>(v)</th>
<th>(\alpha = 0.995)</th>
<th>(\alpha = 0.99)</th>
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</tbody>
</table>

Formulate the hypothesis $H_0: s = \sigma$

Define the test variable ($\chi^2$)

Set the risk $2\alpha$

Calculate $\chi^2_\alpha$ and $\chi^2_{1-\alpha}$

Calculate $\chi^2_{obs} = \frac{(N-1)s^2}{\sigma^2}$

Compare with CI$(1-2\alpha)\%$

If $\chi^2_{obs} > \chi^2_\alpha$ or $\chi^2_{obs} < \chi^2_{1-\alpha}$ reject $H_0$

Formulate the conclusion
Comparison between an experimental and a theoretical variance: unilateral $\chi^2$ test

Formulate the hypothesis $H_0: \sigma \geq \sigma$  
Define the test variable ($\chi^2$)  
Set the risk $\alpha$  
Calculate $\chi^2_{1-\alpha}$  
Calculate $\chi^2_{obs} = \frac{(N-1)s^2}{\sigma^2}$  
Compare with Cl$(1-\alpha)$
If $\chi^2_{obs} < \chi^2_{1-\alpha}$ reject $H_0$
Formulate the conclusion

Formulate the hypothesis $H_0: \sigma \leq \sigma$  
Define the test variable ($\chi^2$)  
Set the risk $\alpha$  
Calculate $\chi^2_{\alpha}$  
Calculate $\chi^2_{obs} = \frac{(N-1)s^2}{\sigma^2}$  
Compare with Cl$(1-\alpha)$
If $\chi^2_{obs} > \chi^2_{\alpha}$ reject $H_0$
Formulate the conclusion

General procedure for a bilateral test

Formulate the hypothesis $H_0$  
Define the test statistics ($q$)  
Set the risk $2\alpha$  
Calculate $q_{\alpha}$ and $q_{1-\alpha}$  
Calculate $q_{obs}$  
Compare with Cl$(1-2\alpha)$
If $q_{obs} > q_{\alpha}$ or $q_{obs} < q_{1-\alpha}$ – reject $H_0$
Formulate the conclusion

for example $\bar{x} = \mu$
or $s_1 = s_2$, etc.
for example $q = z, q = t$, etc

$s$ known?  
y  
N>30?  
y  
Use $z$  
y  
n  
N>30?  
n  
Use $t$ and $s$ for $\sigma$  
y  
y  
Use $z$ and $s$ for $\sigma$  

Reject $H_0$  
$q_{1-\alpha}$  
$q_\alpha$   

6.46
General procedure for a unilateral test

**Formulate the hypothesis** $H_0 \bar{x} \geq \mu$

1. Define the test variable ($q$)
2. Set the risk $\alpha$
3. Calculate $q_{1-\alpha}$
4. Calculate $q_{obs}$
5. Compare with CI$(1-\alpha)$%
   - If $q_{obs} < q_{1-\alpha}$ reject $H_0$

**Formulate the hypothesis** $H_0 \bar{x} \leq \mu$

1. Define the test variable ($q$)
2. Set the risk $\alpha$
3. Calculate $q_{\alpha}$
4. Calculate $q_{obs}$
5. Compare with CI$(1-\alpha)$%
   - If $q_{obs} > q_{\alpha}$ reject $H_0$

**Formulate the conclusion**

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### Key points

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<td>$\sigma_{\text{theoretical}}$ known and $N&gt;30$</td>
<td>$z = \frac{\bar{x} - \mu}{\sigma/\sqrt{N}}$</td>
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<td>Experimental</td>
<td>t-test</td>
<td>$s_{\text{experimental}}$ known or $N&lt;30$</td>
<td>$t = \frac{\bar{x} - \mu}{s/\sqrt{N}}$</td>
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<td>Experimental vs.</td>
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<td>$z$ follows a normal distribution $N_1 \geq 30$ and $N_2 \geq 30$</td>
<td>$z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{N_1} + \frac{\sigma_2^2}{N_2}}}$</td>
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<tr>
<td>Experimental</td>
<td>t-test</td>
<td>$N_1 &lt; 30$ or $N_2 &lt; 30$</td>
<td>$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{N_1} + \frac{s_2^2}{N_2}}}$</td>
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<td>$\chi^2$-test</td>
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<td>$\chi^2_{obs} = (N-1)s^2/\sigma^2$</td>
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<td>Experimental</td>
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<td>$f_{obs} = s^2_1/s^2_2$</td>
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