Exercise 1: Stochastic input

Consider a passive membrane receiving a stochastic input current which is zero for time $t < 0$ and

$$R I(t) = R I_0 + \xi(t), \text{ for } t > 0,$$

where $\xi(t)$ is white noise with

$$\langle \xi(t) \rangle = 0,$$  \hspace{1cm} (2)

$$\langle \xi(t)\xi(t') \rangle = \tau_m a^2 \delta(t-t').$$  \hspace{1cm} (3)

The membrane potential obeys the equation

$$\frac{du}{dt} = -(u - u_{rest}) + RI(t),$$  \hspace{1cm} (4)

with solution

$$u(t) = u_{rest} + \frac{R}{\tau_m} \int_0^t \exp(-s/\tau_m) I(t-s) ds$$  \hspace{1cm} (5)

1.1 Calculate the expected voltage $\langle u(t) \rangle$, where $\langle \rangle$ is the average over multiple repetitions or over a population of neurons having the same dynamics and inputs.

1.2 Calculate the variance of the potential across multiple repetitions: $\text{Var}[u](t) = \langle [u(t) - \langle u(t) \rangle]^2 \rangle$.

Exercise 2: Diffusive noise (stochastic spike arrival)

Consider a passive membrane receiving stochastic synaptic input $S(t) = \sum_f \delta(t - t^f_k)$, where the index $f$ runs over the firing times of a presynaptic neuron. The spike train starts only at $t = 0$, so that $t^f_k > 0$ for all firing times. The membrane potential obeys the equation:

$$\frac{du}{dt} = -\frac{u - u_{rest}}{\tau} + \frac{q}{C} S(t).$$  \hspace{1cm} (6)

where $q$ is the charge brought by each spike and $C$ is the capacitance of the membrane. The solution to this equation writes:

$$u(t) = u_{rest} + \frac{qR}{\tau} \int_0^t \exp(-s/\tau) S(t-s) ds$$  \hspace{1cm} (7)

2.1 Calculate the expected voltage $\langle u(t) \rangle$ as a function of $t$ for a constant presynaptic rate $\langle S(t) \rangle = \nu$ for $t \geq 0$ ($\nu = 0$ for $t < 0$). Where $\langle \rangle$ is the average over multiple repetitions or over a population of
neurons having the same dynamics and inputs.

2.2 Calculate \( \langle u(t)^2 \rangle \). Assume that the spike times of the presynaptic neuron are uncorrelated, i.e., \( \langle S(t)S(t') \rangle = \nu \delta(t-t') + \nu^2 \), and use Eq. 7.

2.3 Calculate the variance of the potential across multiple repetitions: \( Var[u](t) = \langle [u(t) - \langle u(t) \rangle]^2 \rangle \).

Homework:

2.4 Calculate the autocorrelation of the voltage \( \langle u(t)u(t') \rangle \) in the steady state regime (replace the upper bound of the integral by \( \infty \) in equation 7).

2.5 Suppose that there are two presynaptic neurons which fire independently with rates \( \nu_1 = \langle S_1(t) \rangle \) and \( \nu_2 = \langle S_2(t) \rangle \), such that the input to the postynaptic neuron is given by \( w_1S_1(t) + w_2S_2(t) \) where \( w_i \) denote the synaptic weights. Calculate again the mean and autocorrelation of the voltage.

2.6 Redo question 2.5 with correlated spike trains \( S_1(t) = S_2(t) \).

Exercise 3: Firing statistics

Consider a stochastic spike generation process in discrete time. The probability of generating a spike in a time \( \Delta t \) is \( P_{\Delta t} = \nu \Delta t \). Hence when we take the limit of \( \Delta t \) to 0 the expected value of the quantity \( S(t) = \sum_j \delta(t-t_j) \) is:

\[
\langle S(t) \rangle = \lim_{\Delta t \to 0} \frac{P_{\Delta t}(t)}{\Delta t} = \nu ; \text{ for } t > 0.
\]

Consider the probability of having two spikes in different time bins around \( t \) and \( t' \). Define \( \langle S(t)S(t') \rangle \) in a similar fashion, and show that it is equal to \( \nu \delta(t-t') + \nu^2 \).

Exercise 4: Renewal process

We consider a neuron with relative refractoriness. Given an output spike at time \( \hat{t} \), the probability of firing is given by

\[
\rho(t-\hat{t}) = \begin{cases} 
0 & \text{for } t-\hat{t} < t_{\text{abs}} \\
\left|t-\hat{t} - t_{\text{abs}}\right| \rho_0 \frac{\rho_0}{2} & \text{for } t_{\text{abs}} < t-\hat{t} < t_{\text{abs}} + 2 \\
\rho_0 & \text{otherwise.}
\end{cases}
\] (8)

Calculate the survivor function and the interval distribution.