Exercise 1: Stochastic input

Consider a passive membrane receiving a stochastic input current which is zero for time \( t < 0 \) and

\[
RI(t) = RI_0 + \xi(t), \text{ for } t > 0,
\]

where \( \xi(t) \) is white noise with

\[
\langle \xi(t) \rangle = 0,
\]

\[
\langle \xi(t) \xi(t') \rangle = \tau_m a^2 \delta(t - t').
\]

The membrane potential obeys the equation

\[
\frac{du(t)}{dt} = -(u - u_{\text{rest}}) + RI(t),
\]

with solution

\[
u(t) = u_{\text{rest}} + \frac{R}{\tau_m} \int_0^t \exp\left(-\frac{s}{\tau_m}\right)I(t - s)ds
\]

1.1 Calculate the expected voltage \( \langle u(t) \rangle \), where \( \langle \rangle \) is the average over multiple repetitions or over a population of neurons having the same dynamics and inputs.

1.2 Calculate the variance of the potential across multiple repetitions: \( \text{Var}[u(t)] = \langle [u(t) - \langle u(t) \rangle]^2 \rangle \).

Exercise 2: Diffusive noise (stochastic spike arrival)

Consider a passive membrane receiving stochastic synaptic input \( S(t) = \sum_f \delta(t - t_k^f) \), where the index \( f \) runs over the firing times of a presynaptic neuron. The spike train starts only at \( t = 0 \), so that \( t_k^f > 0 \) for all firing times. The membrane potential obeys the equation:

\[
\frac{du(t)}{dt} = -\frac{u - u_{\text{rest}}}{\tau} + \frac{q}{C} S(t).
\]

where \( q \) is the charge brought by each spike and \( C \) is the capacitance of the membrane. The solution to this equation writes:

\[
u(t) = u_{\text{rest}} + \frac{qR}{\tau} \int_0^t \exp(-s/\tau)S(t - s)ds
\]

2.1 Calculate the expected voltage \( \langle u(t) \rangle \) as a function of \( t \) for a constant presynaptic rate \( \langle S(t) \rangle = \nu \) for \( t \geq 0 \) (\( \nu = 0 \) for \( t < 0 \)). Where \( \langle \rangle \) is the average over multiple repetitions or over a population of
neurons having the same dynamics and inputs.

2.2 Calculate $\langle u(t)^2 \rangle$. Assume that the spike times of the presynaptic neuron are uncorrelated, i.e., $\langle S(t)S(t') \rangle = \nu \delta(t-t') + \nu^2$, and use Eq. 7.

2.3 Calculate the variance of the potential across multiple repetitions: $\text{Var}[u(t)] = \langle [u(t) - \langle u(t) \rangle]^2 \rangle$.

**Homework:**

2.4 Calculate the autocorrelation of the voltage $\langle u(t)u(t') \rangle$ in the steady state regime (replace the upper bound of the integral by $\infty$ in equation 7).

2.5 Suppose that there are two presynaptic neurons which fire independently with rates $\nu_1 = \langle S_1(t) \rangle$ and $\nu_2 = \langle S_2(t) \rangle$, such that the input to the postsynaptic neuron is given by $w_1 S_1(t) + w_2 S_2(t)$ where $w_i$ denote the synaptic weights. Calculate again the mean and autocorrelation of the voltage.

2.6 Redo question 2.5 with correlated spike trains $S_1(t) = S_2(t)$.

**Exercise 3: Firing statistics**

Consider a stochastic spike generation process in discrete time. The probability of generating a spike in a time $\Delta t$ is $P_{\Delta t} = \nu \Delta t$. Hence when we take the limit of $\Delta t$ to 0 the expected value of the quantity $S(t) = \sum_f \delta(t - t_f)$ is:

$$\langle S(t) \rangle = \lim_{\Delta t \to 0} \frac{P_{\Delta t}(t)}{\Delta t} = \nu ; \text{ for } t > 0.$$  

Consider the probability of having two spikes in different time bins around $t$ and $t'$. Define $\langle S(t)S(t') \rangle$ in a similar fashion, and show that it is equal to $\nu \delta(t-t') + \nu^2$.

**Exercise 4: Renewal process**

We consider a neuron with relative refractoriness. Given an output spike at time $\hat{t}$, the probability of firing is given by

$$p(t - \hat{t}) = \begin{cases} 0 & \text{for } t - \hat{t} < t_{\text{abs}} \\ \rho_0 \frac{[t - \hat{t} - t_{\text{abs}}]^2}{2} & \text{for } t_{\text{abs}} < t - \hat{t} < t_{\text{abs}} + 2 \\ \rho_0 & \text{otherwise.} \end{cases} \tag{8}$$

Calculate the survivor function and the interval distribution.