DIFFERENTIAL GEOMETRY II

Smooth Manifolds

Author
Linley VION
École Polytechnique Fédérale de Lausanne
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Contents

Chapter 1

Foreword

These lectures notes are based on the Lectures given by Dr. Nikolaos Tsakanikas at EPFL during the Fall semester of 2023-2024 for the course MATH-322 ”Differential geometry II - Smooth manifolds”. To sum up this course, smooth manifolds constitute a certain class of topological spaces which locally look like some Euclidean space $\mathbb{R}^n$ and on which one can do calculus. This course introduces the key concepts of this subject, such as vector fields, differential forms, etc. This course requires a good understanding of mutlivariable and vector calculus, topological and metric spaces. Topology will turn out to be quite useful. The notion of manifold and its equipped objects are recurrent in both mathematics and physics and are of particular interest more globally in geometry.

This course is heavily inspired from John Lee's book on differentiable manifolds [1]. For this reason, most of the proof that do not appear in the lectures notes can be found in that very book. For any person that would lack the prerequisites stated above, the appendices at the end of the book make up a little bit for what is needed but keep in mind it will not be enough. Any student interested into going deeper into the notions presented is invited to consult John Lee's book as it is widely considered as a reference in the domain. The other book considered for that course is Jeffrey Lee's book on Manifolds and Differential Geometry [2] which introduces the notions in a different order. It can be useful to consult both but not required to understand the topics presented during these notes. For complementary reading on differential forms, see [3].

If you find something that may look like a typo or a mistake introduced during the writing process of these notes, it could indeed be a mistake or typo. I apologize in advance if I have let any slide in these notes. Please, do not stay stuck on those and check in case of doubt with Lee's book [1] which should clear any doubts. Throughout these notes, some references are made to the Exercise Sheets for some proofs, exercise, etc. The exercises may have changed or you may not have access to them. In both cases, [1] is again the reference you should turn to.

The last two chapters of these notes were kindly provided by Dr. N. Tsakanikas and concern Multilinear Algebra [4] and Orientations [5] which are referenced in these notes under [Multilinear

Algebra] and [Orientations].

I hope you appreciate this rich and wonderful topic through the reading of these notes!

Chapter 2

Smooth Manifolds

2.1 Definition of a topological manifold

Definition 2.1. a topological manifolds of dimension n (or top. n-manifold) is a topological space M with the following properties:

(i) M is a Hausdorff space: for every pair of distinct points p, $q \in M$, there are disjoints open sets U, $V \subseteq M$ s.t $p \in U$ and $q \in V$.

(ii) M is second-countable: there is a countable basis for the topology of M.

(iii) M is locally Euclidean of dimension n : each pt of M has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$; that is, for each $p \in M$ we can find:

  (a) an open subset $U \subseteq M$ containing p

  (b) an open subset $\hat{U} \subseteq \mathbb{R}^n$ and

  (c) a homeomorphism $\varphi : U \to \hat{U}$.

Comments

1. Every topological manifold has, by definition, a specific, well-defined dimension. In fact, it can be shown (using De Rham cohomology) that the dimension of a (non-empty) top. manifold is a topological invariant: A non-empty topological n-manifold cannot be homeomorphic to a topological m-manifold unless $m = n$.

2. The conditions in Definition 2.1 ensure that manifolds behave in the way we expect from our experience with Euclidean spaces.

3. The lines with two origins (see Exercise Sheet 1) is locally Euclidean, second-countable, but not Hausdorff.

   A disjoint union of uncountably many copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second-countable.

2.2 Charts

Definition 2.2. Let M be a topological n-manifold. A coordinate chart on M is a pair (U, φ), where U is an open subset of M and $\varphi : U \to \hat{U}$ is a homeomorphism from U to an open subset $\hat{U} \subseteq \mathbb{R}^n$. The set U is called a coordinate domain, or a coordinate neighborhood of each of its points. The map φ is called a (local) coordinate map and its components functions $(x^1, \dots, x^n)$, defined by $\varphi(p) = (x^1(p), \dots, x^n(p))$, are called local coordinates on U.

By definition of a topological manifold, each point $p \in M$ is contained in the of some chart (U, φ). If $\varphi(p) = 0$, then we say that the chart is centered at p. See also coordinate ball and coordinate cube in Lee's book [1].

Example 2.3.

0. The basic example of a topological n-manifold is $\mathbb{R}^n$ itself. It is Hausdorff, because it is a metric space, and it is second-countable, because a collection of all open balls with rational centers and rational radii is a countable basis for its topology.

   Moreover, every open subset of a topological n-manifold is a topological n-manifold (with the subspace topology), because the Hausdorff and second-countability properties are inherited by subspaces.

1. Graph of continuous functions: Let $U \subseteq \mathbb{R}^n$ be an open subset and let $f : U \to \mathbb{R}^k$ be a continuous function. The graph of f is the subset

$$\Gamma(f) := \Big\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^k \mid x \in U,\ y = f(x) \Big\}$$

   with the subspace topology. Let $\pi_1 = \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ be the projection onto the first factor and let $\varphi : \Gamma(f) \in U$ be the restriction of π_1 to $\Gamma(f)$: $\varphi(x, y) = x$, $(x, y) \in \Gamma(f)$. Since φ is the restriction of a continuous map, it is continuous; and it is a homeomorphism, because it has a continuous inverse given by $\varphi^{-1}(x) = (x, f(x))$. Thus $\Gamma(f)$ is a topological manifold of dimension n. In fact, $\Gamma(f)$ is homeomorphic to U itself, and $(\Gamma(f), \varphi)$ is a global coordinate chart, called the graph coordinates.

   The same observation applies to any subset of $\mathbb{R}^{n+k}$ defined by setting any k of the coordinates (not necessarily the last k) equal to some continuous function of the other n, which are restricted to lie in an open subset pf $\mathbb{R}^n$.

2. Spheres: For each integer $n \geq 0$, the unit n-sphere is the subset

$$\mathbb{S}^n := \left\{x \in \mathbb{R}^{n+1} \mid |x| = 1\right\} \subseteq \mathbb{R}^{n+1}$$

It is Hausdorff and second-countable, because it is a topological subspace of $\mathbb{R}^{n+1}$. To show that it is locally Euclidean, for each $i \in \{1, \dots, n+1\}$, consider

$$U_i^+ := \left\{(x^1, \dots, x^{n+1}) \in \mathbb{R}^{n+1} \mid x^i > 0\right\}$$
$$U_i^- := \left\{(x^1, \dots, x^{n+1}) \in \mathbb{R}^{n+1} \mid x^i < 0\right\}$$

Let $f : \mathbb{B} := \{x \in \mathbb{R}^n \mid |x| < 1\} \to \mathbb{R}$ be the continuous function $f(u) = \sqrt{1 - |u|^2}$. Then for each $i \in \{1, \dots, n+1\}$ it is easy to check that $U_i^+ \cap \mathbb{S}^n$ is the graph of the function

$$x^i = f(x^1, \dots \underbrace{, \hat{x}^i,}_{\text{omitted}} \dots, x^{n+1})$$

and that $U_i^- \cap \mathbb{S}^n$ is the graph of the function

$$x^i = -f(x^1, \dots \underbrace{, \hat{x}^i,}_{\text{omitted}} \dots, x^{n+1}).$$

Thus each subset $U_i^\pm \cap \mathbb{S}^n$ is locally Euclidean of dimension n, and the maps

$$\varphi_i^\pm : \quad U_i^\pm \cap \mathbb{S}^n \to \mathbb{B}^n \quad (x^1, \dots, x^{n+1}) \mapsto (x^1, \dots, \hat{x}^i, \dots, x^{n+1})$$

are graph coordinates for $\mathbb{S}^n$. Since each point of $\mathbb{S}^n$ is in the domain of at least one of these $2n+2$ charts, we conclude that $\mathbb{S}^n$ is a topological n-manifold.

Topological manifolds :

– Suitable for the study of topological properties (e.g compactness, connectedness, etc).

– Not suitable for calculus : being "differentiable" is not invariant under homeomorphisms.

⇒ To make sense of derivatives of maps between manifolds, we need to introduce a new kind of manifold; it will be a topological manifold with some extra structure which will allow us to decide which maps are smooth.

2.3 Smooth charts, atlases and smooth structures

Definition 2.4. Let M be a topological manifold. If (U, φ) and (V, ψ) are two charts such that $U \cap V \neq \varnothing$, then the composite map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

is called the transition map from φ to ψ (it is a homeomorphism).

Two charts (U, φ) and (V, ψ) are said to be smoothly compatible if either $U \cap V = \varnothing$ or the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism (i.e smooth and bijective with a smooth inverse). Since $\varphi(U \cap V)$ and $\psi(U \cap V)$ are open subsets of $\mathbb{R}^n$, smoothness of this map is to be interpreted in the ordinary sense of having continuous partial derivatives at all orders (here diffeomorphisms are C^∞ maps).

An atlas for M is a collection of charts whose domains cover M. An atlas $\mathcal{A}$ is called a smooth atlas if any two charts in $\mathcal{A}$ are smoothly compatible.

Remark To show that an atlas is smooth, we need only to verify that each transition map $\psi \circ \varphi^{-1}$ is smooth whenever (U, φ) and (V, ψ) are charts in $\mathcal{A}$; once we have proved this, it follows that $\psi \circ \varphi^{-1}$ is a diffeomorphism because its inverse $\varphi \circ \psi^{-1} = \left(\psi \circ \varphi^{-1}\right)^{-1}$ is one of the transition maps we have already shown to be smooth. Alternatively, given two particular charts (U, φ) and (V, ψ), it is often easier to show that they are smoothly compatible by verifying that $\psi \circ \varphi^{-1}$ is smooth with non-singular Jacobian at each point (consequence of Inverse Function Theorem).

Definition 2.3 (continued) : A smooth atlas $\mathcal{A}$ on a topological manifold M is called maximal (or complete) if it is not properly contained in any larger smooth atlas. This just means that any chart which is smoothly compatible with every chart is $\mathcal{A}$ is already in $\mathcal{A}$.

Definition 2.5. Let M be a topological manifold. A smooth structure on M is a maximal smooth atlas. A smooth manifold is a pair $(M, \mathcal{A})$, where M is a topological manifold and $\mathcal{A}$ is a smooth structure on M.

Remark : A smooth structure is an additional piece of data that must be added to a topological manifold before we are entitled to talk about a ”smooth manifold”.

A given topological manifold may have many smooth structures (in fact, if it has one, then it has infinitely many) but it may also have no smooth structure at all.

It is in general not convenient to define a smooth structure by explicitly describing a maximal smooth atlas, because such atlas contains very many charts. The next result shows that we need only to specify a smooth atlas.

Proposition 2.6. Let M be a topological manifold.

a) Every smooth atlas $\mathcal{A}$ for M is contained in a unique maximal smooth atlas, called the smooth structure determined by $\mathcal{A}$.

b) Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.

Proof. See Exercise Sheet 2. $\square$

For example, if a topological manifold M can be covered by a single chart, then the smooth compatibility condition is trivially satisfied, so any such chart determines automatically a smooth structure on M.

Definition 2.7. Let M be a smooth manifold. Any chart (U, φ) contained in the maximal smooth atlas is called a smooth chart. The corresponding coordinate map φ is called a smooth coordinate map, and its domain U is called a smooth coordinate domain, or smooth coordinate neighborhood of each of its points.

Example 2.8. 0. For each $n \in \mathbb{N}$, the Euclidean space $\mathbb{R}^n$ is a smooth n-manifold with the smooth structure determined by the atlas $\{(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})\}$. We call this the standard smooth structure on $\mathbb{R}^n$ and the resulting coordinate map, standard coordinates. With respect to this smooth structure, the smooth coordinate charts for $\mathbb{R}^n$ are exactly those charts (U, φ) such that φ is a diffeomorphism (in the usual sense) from $U \subseteq \mathbb{R}^n$ to another subset $\hat{U} \subseteq \mathbb{R}^n$.

1. Graphs of smooth functions : If $U \subseteq \mathbb{R}^n$ is an open subset and if $f : U \to \mathbb{R}^k$ is a smooth function, then the graph $\Gamma(f)$ of f is a topological $n-$manifold in the subspace topology. Since $\Gamma(f)$ is covered by the single graph coordinate chart $\varphi : \Gamma(f) \to U$, we can put a canonical smooth structure on $\Gamma(f)$ by declaring $(\Gamma(f), \varphi)$ to be a smooth chart.

2. Spheres : The n-sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ is a topological n-manifold. We put a smooth structure on $\mathbb{S}^n$ as follows. For each $i \in \{1, \dots, n+1\}$, we consider the graph coordinate charts $(U_i^{\pm} \cap \mathbb{S}^n, \varphi_i^{\pm})$. For any $i \neq j$ and any choice of $\pm$ signs, the transition map $\varphi_i^{\pm} \circ (\varphi_j^{\pm})^{-1}$ and $\varphi_i^{\pm} \circ (\varphi_j^{\mp})^{-1}$ are easily computed. For example, when $i < j$, we get

$$\begin{aligned} \varphi_i^{+} \circ (\varphi_j^{+})^{-1}(u^1, \dots, u^n) &= \varphi_i^{+}(u^1, \dots, \sqrt{1-u^2}, \dots, u^n) \\ &= (u^1, \dots, \hat{u}^i, \dots, \underbrace{\sqrt{1-u^2}}_{j\text{-th}}, \dots, u^n) \end{aligned}$$

and similar formula holds in the other cases. When $i = j$, the domains of φ_i^{+} and φ_i^{-} are disjoints, so there is nothing to check. Thus, the collection of charts $\{(U_i^{\pm} \cap \mathbb{S}^n, \varphi_i^{\pm})\}_{i=1}^{n+1}$ is a smooth atlas, so it defines a smooth structure on $\mathbb{S}^n$, which we call its standard smooth structure.

3. Open Submanifolds : Let U be any open subset of $\mathbb{R}^n$. The, U is a topological n-manifold, and the single chart (U, Id_U) determines a smooth structure on U.
More generally, let M be a smooth n-manifold and let $U \subseteq M$ be an open subset. Define an

atlas on U by

$$\mathcal{A}_U = \{\text{smooth charts } (V, \varphi) \text{ for } M \text{ such that } V \subseteq U\}.$$

Every point $p \in U$ is contained in the domain of some chart $W, \varphi)$ for M; if we set $V = W \cap U$, then $(V, \varphi|_V)$ is a chart in $\mathcal{A}_U$ whose domain contains p. Therefore, U is covered by the domains of the charts in $\mathcal{A}_U$, and it is easy to verify that this is a smooth atlas for U.

Thus, any open subset of M is itself a smooth n-manifold in a natural way. Endowed with the smooth structure, we call any open subset an open submanifold of M.

In the examples we have seen so far, we constructed a smooth manifold structure in two stages : we started wit a topological space and checked that it was a topological manifold, and then we specified a smooth structure. The following lemma shows how, given a set of suitable ”charts” that overlap smoothly, we can use the charts to define bot a topology and a smooth structure on the set.

Lemma 2.9. Let M be a set. Suppose we are given a collection $\{\mathcal{U}_\alpha\}$ of subsets of M together with maps $\varphi_\alpha : \mathcal{U}_\alpha \to \mathbb{R}^n$ such that the following properties are satisfied:

(i) For each α, φ_α is a bijection between $\mathcal{U}_\alpha$ and an open subset $\varphi_\alpha(\mathcal{U}_\alpha) \subseteq \mathbb{R}^n$.

(ii) For each α and β, the sets $\varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ and $\varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ are open in $\mathbb{R}^n$.

(iii) Whenever $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$, the map

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \to \varphi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$$

is smooth.

(iv) Countably many of the sets $\mathcal{U}_\alpha$ cover M.

(v) Whenever $p, q \in M$ with $p \neq q$, either there exists some $\mathcal{U}_\alpha$ containing both p and q or there exist disjoint sets $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ with $p \in \mathcal{U}_\alpha$ and $q \in \mathcal{U}_\beta$.

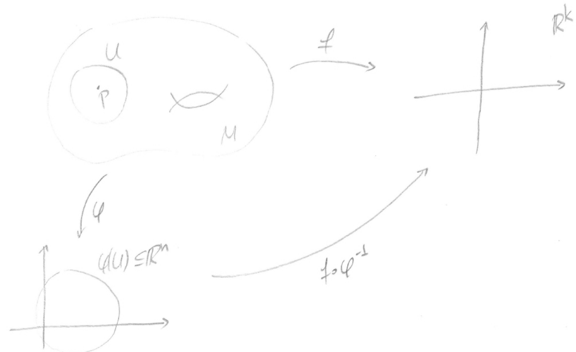
Then M has a unique manifold structure such that each $(\mathcal{U}_\alpha, \varphi_\alpha)$ is a smooth chart.

Proof. Details of the proof: Lee, Lemma 1.35 [1]. Key idea: define the topology on M by taking all the sets of the form $\varphi_\alpha^{-1}(V)$, $V \subseteq \mathbb{R}^n$ open, as a basis. □

Chapter 3

Smooth Maps

Definition 3.1. Let M be a smooth n-manifold and let $f : M \to \mathbb{R}^k$ be a function, where $k \geq n$. We say that f is a smooth map if for every point $p \in M$ there exists a smooth chart (U, φ) for M such that $p \in U$ and $f \circ \varphi^{-1}$ is smooth on the open subset $\varphi(U) \subseteq \mathbb{R}^n$.



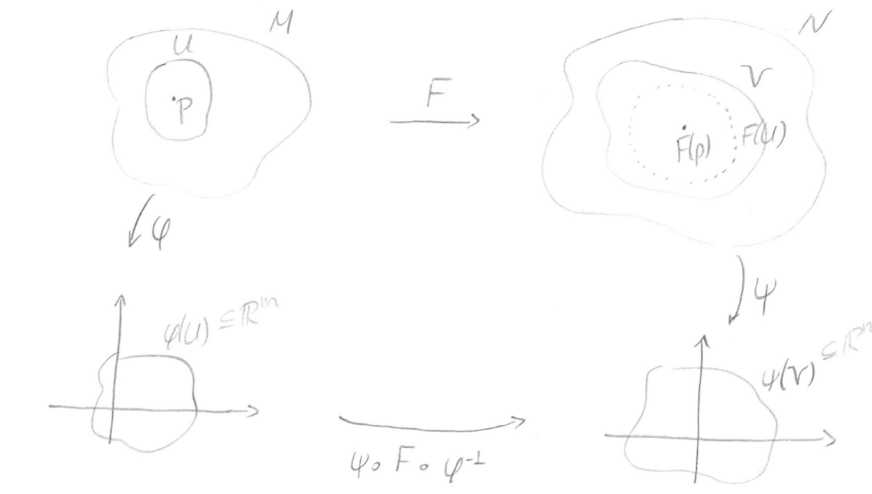
Remark :

1. If M is a smooth manifold and $f : M \to \mathbb{R}^k$ is a smooth map, then $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^k$ is smooth for every smooth chart (U, φ) for M (see Example 5.3).

2. Let M be a smooth manifold. The set $C^\infty(M)$ of all smooth real-valued functions on M is an infinite-dimensional $\mathbb{R}$-vector space: sums and constant multiples of smooth functions are smooth (see also Exercise Sheet 3). Moreover, pointwise multiplication turns $C^\infty(M)$ into a commutative ring and a commutative and associative $\mathbb{R}$-algebra.

Definition 3.2. Let M be a smooth manifold. Given a function $f : M \to \mathbb{R}^k$ and a chart (U, φ) for M, the function $\hat{f} = f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^k$ is called the coordinate representation of f. By definition, the function f is smooth if and only if its coordinate representation is smooth in the same smooth chart around each point. By the previous Remark, smooth functions have smooth coordinate representations in every smooth chart.

Definition 3.3. Let $F : M \to N$ be a map between smooth manifolds. We say that F is a smooth map if for every $p \in M$, there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subseteq V$ and the map $\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is smooth.



Observe that Definition 3.1 is a special case of Definition 3.3 by taking $N = V = \mathbb{R}^k$ and $\psi = \mathrm{Id}_{\mathbb{R}^k}$.

Proposition 3.4. Every smooth map is continuous.

Proof. Let $F : M \to N$ be a smooth map between smooth manifolds. Fix $p \in M$. Since F is smooth, there are smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subseteq V$ and $\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is smooth, and hence continuous. Since $F(U) \subseteq V$ and the maps φ and ψ are homeomorphisms, the map

$$F|_U = \psi^{-1} \circ (\psi \circ F \circ \varphi^{-1}) \circ \varphi : U \to V$$

is continuous as a composition of continuous maps. Hence, F is continuous in a neighborhood of each point, and thus continuous on M. $\square$

Comment The requirement that "$\forall p \in M\, \exists (U, \varphi) \ni p\, \exists (V, \psi) \ni F(p)\, \text{s.t.}\, F(U) \subseteq V$" in the definition of smoothness is included precisely so that smoothness implies continuity.

Definition 3.5. Let $F : M \to N$ be a smooth map between smooth manifolds. If (U, φ) and (V, ψ) are smooth charts for M and N, respectively, then we call $\hat{F} = \psi \circ F \circ \varphi^{-1}$ the coordinate representation of F with respect to the given coordinates. It maps $\varphi(U \cap F^{-1}(V))$ to $\psi(V)$.

Remark Let $F : M \to N$ be a smooth map between smooth manifolds. Then the coordinate representation of F with respect to every pair of smooth charts for M and N is smooth (see Example 5.3).

- There are equivalent characterizations of smoothness (see Example 5.3). For example, a map $F : M \to N$ between smooth manifolds is smooth if and only if F is continuous, and there exist smooth atlases $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ for M and N, respectively, such that for each α and β, $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is a smooth map from $\varphi_\alpha(U_\alpha \cap F^{-1}(V_\beta))$ to $\psi_\beta(V_\beta)$.

- Smoothness is local (see Example 5.3); if $F : M \to N$ is a map between smooth manifolds and if $\forall p \in M \, \exists U \ni p$ such that $F|_U$ is smooth, then F is smooth.

- **Gluing Lemma for Smooth Maps:** Let M and N be smooth manifolds and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover for M. Suppose that for each $\alpha \in A$ we are given a smooth map $F_\alpha : U_\alpha \to N$ such that the maps agree on overlaps: $F_\alpha|_{U_\alpha \cap U_\beta} = F_\beta|_{U_\alpha \cap U_\beta}$ for all $\alpha, \beta \in A$. Then there exists a unique smooth map $F : M \to N$ such that $F|_{U_\alpha} = F_\alpha$ for each $\alpha \in A$.

Proposition 3.6. Let M, N, and P be smooth manifolds.

(a) Every constant map $c : M \to N$ is smooth.

(b) The identity map Id_M of M is smooth.

(c) If $U \subseteq M$ is an open submanifold, then the inclusion map $\iota : U \hookrightarrow M$ is smooth.

(d) If $F : M \to N$ and $G : N \to P$ are smooth, then so is $G \circ F : M \to P$.

Proof. See Exercise Sheet 3. □

Example 3.7. Consider the unit n-sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ with its standard smooth structure. The inclusion map $\iota : \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ is continuous (inclusion map of topological subspace). It is a smooth map, because its coordinate representation with respect to any of the graph coordinates of Example 1.3(2) is

$$\begin{aligned}\hat{\iota}(u^1, \dots, u^n) &= \iota \circ (\varphi_i^{\pm})^{-1}(u^1, \dots, u^n) \\ &= \left(u^1, \dots, u^{i-1}, \pm\sqrt{1 - |u|^2}, u^i, \dots, u^n\right),\end{aligned}$$

which is smooth on its domain (the set where $|u| < 1$).

Definition 3.8. Let M and N be smooth manifolds.

- A <u>diffeomorphism</u> from M to N is a smooth bijective map $M \to N$ that has a smooth inverse.

- We say that M and N are <u>diffeomorphic</u> if there exists a diffeomorphism between them.

Example 3.9. 1) Consider the maps

$$F : \mathbb{B}^n \to \mathbb{R}^n, \quad x \mapsto \frac{x}{\sqrt{1 - |x|^2}}$$

and

$$G : \mathbb{R}^n \to \mathbb{B}^n, \quad y \mapsto \frac{y}{\sqrt{1 + |y|^2}}$$

These maps are smooth, and it is straightforward to check that they are inverses of each other. Thus, they are both diffeomorphisms, so $\mathbb{B}^n \cong \mathbb{R}^n$.

2) If M is a smooth manifold and if (U, φ) is a smooth chart on M, then $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ is a diffeomorphism. (In fact, it has an identity map as a coordinate representation.)

Proposition 3.10. (Properties of Diffeomorphisms)

(a) Every composition of diffeomorphisms is a diffeomorphism.

(b) Every finite product of diffeomorphisms between smooth manifolds is a diffeomorphism.

(c) Every diffeomorphism is a homeomorphism and an open map.

(d) The restriction of a diffeomorphism to an open submanifold is a diffeomorphism onto its image.

(e) "Diffeomorphic" is an equivalence relation on the class of all smooth manifolds.

Proof. Exercises! □

Just as two topological spaces are considered to be "the same" if they are homeomorphic, two smooth manifolds are essentially indistinguishable if they are diffeomorphic. The central concern of smooth manifold theory is the study of properties of smooth manifolds that are preserved by diffeomorphisms. The dimension is one such property: a non-empty smooth N-manifold cannot be diffeomorphic to a non-empty smooth M-manifold unless $m = n$. (This is a consequence of the chain rule.)

Chapter 4

Partition of Unity

We now discuss partitions of unity, which are tools for ”blending together” local smooth objects into global ones without necessarily assuming that they agree on overlaps (cf. p.18, gluing lemma). They are indispensable in smooth manifold theory, and we will see later some first applications of partitions of unity.

4.1 Partition of Unity: definition and existence.

Definition 4.1. Let M be a topological space and let $f : M \to \mathbb{R}^k$ be a function. The support of f is defined as

$$\operatorname{supp} f = \overline{\{p \in M \mid f(p) \neq 0\}}$$

- If $\operatorname{supp} f$ is contained in some open subset $U \subseteq M$, we say that f is supported in U.
- If $\operatorname{supp} f$ is a compact set (e.g., if M is a compact space), we say that f is compactly supported.

Definition 4.2. Let M be a topological space and let $\mathcal{X} = (X_\alpha)_{\alpha \in A}$ be an open cover of M, indexed by a set A. A partition of unity subordinate to $\mathcal{X}$ is an indexed family $\{\psi_\alpha\}_{\alpha \in A}$ of continuous functions $\psi_\alpha : M \to \mathbb{R}$ with the following properties:

(i) $0 \leq \psi_\alpha(x) \leq 1$, for all $\alpha \in A$ and $x \in M$.

(ii) $\operatorname{supp} \psi_\alpha \subseteq X_\alpha$, for all $\alpha \in A$.

(iii) The family of supports $\{\operatorname{supp} \psi_\alpha\}_{\alpha \in A}$ is locally finite, i.e., every point has a neighborhood that intersects $\operatorname{supp} \psi_\alpha$ for only finitely many values of α.

(iv) $\sum_{\alpha \in A} \psi_\alpha(x) = 1$, for all $x \in M$.

Due to the local finiteness condition (iii), the sum in (iv) has only finitely many non-zero terms in a neighborhood of each point, so there is no issue of convergence.

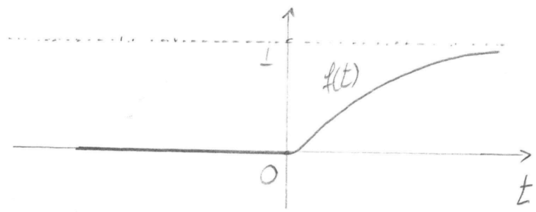
If M is a smooth manifold in Definition 2.19, then a smooth partition of unity is one for which each of the functions φ_α is smooth.

Theorem 4.3. (Existence of smooth partitions of unity): Let M be a smooth manifold and let $\mathcal{X} = (X_\alpha)_{\alpha \in A}$ be an open cover of M. Then there exists a smooth partition of unity subordinate to $\mathcal{X}$.

For a detailed proof of Theorem 4.3 we refer to [Lee [1], Theorem 2.23]. We will only review the main ingredients for the proof of Theorem 4.3:

1. **Inputs from analysis**

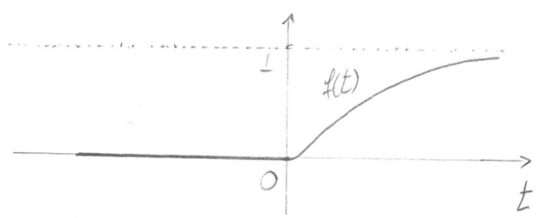
- The function $f : \mathbb{R} \to \mathbb{R}, \quad f(t) = \begin{cases} e^{-\frac{1}{t}}, & t > 0 \\ 0, & t \leq 0 \end{cases}$



  is smooth. [Lee [1], Lemma 2.21]

- **Existence of cutoff functions**: Given $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$, there exists a smooth function $h : \mathbb{R} \to \mathbb{R}$ such that:

$$\begin{aligned} h(t) &\equiv 1 \quad \text{for } t \leq r_1 \\ 0 < h(t) &< 1 \quad \text{for } r_1 < t < r_2 \\ h(t) &\equiv 0 \quad \text{for } t \geq r_2 \end{aligned}$$



  (e.g., take $h(t) := \frac{f(r_2 - t)}{f(r_2 - t) + f(t - r_1)}$, where f is as above)

- Given $r_1, r_2 \in \mathbb{R}$ with $0 < r_1 < r_2$, there exists a smooth function $H : \mathbb{R}^n \to \mathbb{R}$ such that:

$$\begin{aligned} H &\equiv 1 \quad \text{on } \overline{B_{r_1}(0)} \\ 0 < H(x) &< 1 \quad \text{for } x \in B_{r_2}(0) \setminus \overline{B_{r_1}(0)} \\ H &\equiv 0 \quad \text{on } \mathbb{R}^n \setminus B_{r_2}(0) \end{aligned}$$

  (e.g., take $H(x) = h(\|x\|)$, where h is as above)

2. **Inputs from topology**

- **Paracompactness** - one of the main reasons why second countability is included in the definition of topological manifolds.
- Let M be a topological space. A collection $\mathcal{X}$ of subsets of M is called locally finite if each point of M has a neighborhood that intersects at most finitely many of the sets in $\mathcal{X}$. Given a cover $\mathcal{U}$ of M, another cover $\mathcal{V}$ is called a refinement of $\mathcal{U}$ if for each $V \in \mathcal{V}$ there exists some $U \in \mathcal{U}$ such that $V \subseteq U$. We say that M is paracompact if every open cover of M admits an open, locally finite refinement.
- (Manifolds are paracompact): Every topological manifold is paracompact. In fact, given a topological manifold M, an open cover $\mathcal{X}$ of M and any basis $\mathcal{B}$ for the topology of M, there exists a countable, locally finite, open refinement of $\mathcal{X}$ consisting of elements of $\mathcal{B}$. [Lee [1], Theorem 1.15]

4.2 Applications of Partitions of Unity

Finally, we present some applications of partitions of unity.

1. **Existence of smooth bump functions**
   If M is a topological space, $A \subseteq M$ is a closed subset and $U \subseteq M$ is an open subset such that $A \subseteq U$, a continuous function $\psi : M \to \mathbb{R}$ is called a bump function for A supported in U if

$$\begin{aligned} 0 \leq \psi(x) \leq 1, \quad \forall x \in M \\ \psi \equiv 1 \text{ on } A \\ \operatorname{supp} \psi \subseteq U \end{aligned}$$

Proposition 4.4. Let M be a smooth manifold. For any closed subset $A \subseteq M$ and any open subset $U \subseteq M$ containing A, there exist a smooth bump function for A supported in U.

Proof. Set $U_0 := U$ and $U_1 = M \backslash A$, and let $\{\psi_0, \psi_1\}$ be a partition of unity subordinate to the open cover $\{U_0, U_1\}$ of M. Since $\psi_1 \equiv 0$ on A, and $\sum \psi_i \equiv 1$ on A, the function ψ_0 has the requested properties. □

2. **Extension Lemma for smooth functions**
   Let M and N be smooth manifolds and let $A \subseteq M$ be an arbitrary subset. We say a map $F : A \to N$ is smooth on A if it has a smooth extension in a neighborhood of each point; namely, for every point in A there exists a an open subset $p \in W \subseteq M$ and a smooth map $\tilde{F} : W \to N$ whose restriction to $W \cap A$ agrees with A.

Lemma 4.5. Let M be a smooth manifold, $A \subseteq M$ a closed subset and $f : A \to \mathbb{R}^k$ a smooth function. For any open subset $U \subseteq M$ containing A, there exists a smooth function $\tilde{f} : U \to \mathbb{R}^k$ such that $\tilde{f}|_A = f$ and $\operatorname{supp} \tilde{f} \subseteq U$.

Proof. For each $p \in A$, choose a neighborhood W_p of p and a smooth function $\tilde{f}_p : W_p \to \mathbb{R}^k$ such that $\tilde{f}_p|_{W_p \cap A} = f$. Replacing W_p by $W_p \cap U$, w.m.a.t $W_p \subseteq U$. The family of sets $\{W_p\}_{p \in A} \cup \{M \backslash A\}$ is an open cover of M. Let $\{\psi_p\}_{p \in A} \cup \{\psi_0\}$ be a smooth of unity subordinate to this cover, with supp $\psi_p \subseteq W_p$ and supp $\psi_0 \subseteq M \backslash A$.

For each $p \in A$, the product $\psi_p \tilde{f}_p$ is smooth on W_p, and has a smooth extension to all of M if we interpret it to be zero on $M \backslash \text{supp } \psi_p$. Thus, we can define the function

$$\tilde{f} : M \to \mathbb{R}^k, \quad x \mapsto \sum_{p \in A} \psi_p(x) \tilde{f}_p(x)$$

Since the collection of supports $\{\text{supp } \psi_p\}_{p \in A}$ is locally finite, this sum has actually finitely many non-zero terms in a neighborhood of each point of M, and therefore defines a smooth function. If $x \in A$, then $\psi_0(x) = 0$ and $\tilde{f}_p(x) = f(x)$ for each p such that $\psi_p(x) \neq 0$, so

$$\tilde{f}(x) = \sum_{p \in A} \psi_p(x) \tilde{f}_p(x) = \left(\psi_0(x) + \sum_{p \in A} \psi_p(x) \right) f(x) = f(x)$$

Thus, $\tilde{f}$ is indeed an extension of f. Finally we have

$$\text{supp } \tilde{f} \subseteq \overline{\bigcup_{p \in A} \text{supp } \psi_p} = \bigcup_{p \in A} \text{supp } \psi_p \subseteq U$$

property of locally finite collections, see [Lee [1], Lemma 1.13]. □

Comments:

i) The conclusion of the extension lemma can be false if A is not closed.

ii) The assumption in the extension lemma that the codomain is $\mathbb{R}^k$, and not some other manifold, is necessary (for other codomains, extensions can fail to exist for topological reasons).

3. **Existence of smooth exhaustion functions** [Lee [1], Prop 2.23]

4. **Level Sets of Smooth Functions:** Let M be a smooth manifold. If K is a closed subset of M, then there exists a smooth non-negative function $f : M \to \mathbb{R}$ such that $f^{-1}(0) = K$. [Lee [1], Theorem 2.29]

Chapter 5

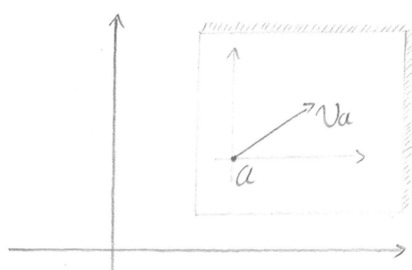
The Tangent bundle

5.1 Geometric Tangent Space

Given a point $a \in \mathbb{R}^n$, we define the geometric tangent space to $\mathbb{R}^n$ at a to be the set

$$\mathbb{R}^n_a := \{a\} \times \mathbb{R}^n = \{(a, v) \mid v \in \mathbb{R}^n\}.$$

We abbreviate (a, v) as v_a or $v|_a$, and we think of v_a as the vector v with its initial point at a.



The set $\mathbb{R}^n_a$ is an $\mathbb{R}$-vector space under the natural operation

$$v_a + w_a := (v + w)_a, \quad \lambda v_a = (\lambda v)_a,$$

and the vectors $e_i|_a, 1 \leq i \leq n$ (where e_i denotes the i-th standard basis vector) are a basis of $\mathbb{R}^n_a$. In fact, $\mathbb{R}^n_a$ is essentially the same as $\mathbb{R}^n$ itself; the only reason we add the index a is so that the geometric tangent spaces $\mathbb{R}^n_a$ and $\mathbb{R}^n_b$ at distinct points a and b are disjoint sets.

Definition 5.1. Given $a \in \mathbb{R}^n$, a map $w : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is called a derivation at a if it is $\mathbb{R}$-linear and satisfies the product rule:

$$w(fg) = f(a)w(g) + g(a)w(f).$$

We denote by $T_a\mathbb{R}^n$ the set of all derivations of $C^\infty(\mathbb{R}^n)$ at a. Clearly, $T_a\mathbb{R}^n$ is an $\mathbb{R}$-vector space under the operations

$$(w_1 + w_2)f = w_1 f + w_2 f, \quad (\lambda w)f = \lambda w f.$$

The most important fact about $T_a\mathbb{R}^n$ is that it is finite-dimensional; in fact, it is naturally isomorphic to the geometric tangent space $\mathbb{R}^n_a$ that we defined above, based on the following lemma.

Lemma 5.2. Let $a \in \mathbb{R}^n$, $w \in T_a\mathbb{R}^n$, and $f, g \in C^\infty(\mathbb{R}^n)$.

(a) If f is constant, then $wf = 0$.

(b) If $f(a) = g(a) = 0$, then $w(fg) = 0$.

Proof. (a) Consider the constant function $f_1 \equiv 1 \in C^\infty(\mathbb{R}^n)$. We have

$$w(f_1) = w(f_1 \cdot f_1) = f_1(a)w(f_1) + f_1(a)w(f_1) \implies w(f_1) = 0.$$

Since $f \equiv c$ is constant, we obtain

$$w(f) = w(cf_1) = cw(f_1) = 0.$$

(b) Follows immediately from the product rule.

□

Proposition 5.3. Let $a \in \mathbb{R}^n$.

(a) For each geometric tangent vector $v_a \in \mathbb{R}^n_a$, the map

$$\begin{aligned} D_v|_a : C^\infty(\mathbb{R}^n) &\to \mathbb{R} \\ f &\mapsto D_v|_a f = D_v f(a) = \left.\frac{d}{dt}\right|_{t=0} f(a + tv) \end{aligned}$$

(directional derivative of f in the direction of v at a) is a derivation of $C^\infty(\mathbb{R}^n)$ at a.

(b) The map

$$\begin{aligned} \Phi : \mathbb{R}^n_a &\to T_a\mathbb{R}^n \\ v &\mapsto D_v|_a \end{aligned}$$

is an $\mathbb{R}$-linear isomorphism.

(c) The n derivations

$$\left.\frac{\partial}{\partial x^1}\right|_a, \ldots, \left.\frac{\partial}{\partial x^n}\right|_a$$

defined by

$$\left.\frac{\partial}{\partial x^i}\right|_a f = \frac{\partial f}{\partial x^i}(a), \quad 1 \le i \le n,$$

form a basis of $T_a\mathbb{R}^n$, and thus

$$\dim_{\mathbb{R}} T_a\mathbb{R}^n = n.$$

Proof. (a) Easy to check (using calculus).

(b)
- Linearity: For every $f \in C^\infty(\mathbb{R}^n)$ we have

$$\begin{aligned}
\Phi(\lambda_1 v_1 + \lambda_2 v_2)(f) &= D_{\lambda_1 v_1 + \lambda_2 v_2}|_a f \\
&= \frac{d}{dt}\Big|_{t=0} f(a + t(\lambda_1 v_1 + \lambda_2 v_2)) \\
&= Df(a) \cdot (\lambda_1 v_1 + \lambda_2 v_2) \\
&= \lambda_1 \frac{d}{dt}\Big|_{t=0} f(a + tv_1) + \lambda_2 \frac{d}{dt}\Big|_{t=0} f(a + tv_2) \\
&= \lambda_1 \Phi(v_1)(f) + \lambda_2 \Phi(v_2)(f)
\end{aligned}$$

  which shows the $\mathbb{R}$-linearity of Φ.

- Injectivity: Suppose that $\Phi(v_a) = D_v|_a = 0$ is the zero derivation. Writing $v_a = v^i e_i|_a$ in terms of the standard basis, and considering the j-th coordinate function $x^j : \mathbb{R}^n \to \mathbb{R}$ thought of as a smooth function on $\mathbb{R}^n$, we obtain

$$0 = D_v|_a x^j = v^i \frac{\partial}{\partial x^i}(x^j)|_{x=a} = v^i \delta_i^j = v^j$$

  where the last equality follows because $\frac{\partial x^j}{\partial x^i} = 0$, $i \neq j$ and $\frac{\partial x^i}{\partial x^i} = 1$. Hence, $v_a = 0 \in \mathbb{R}^n$.

- Surjectivity: Let $w \in T_a\mathbb{R}^n$. Set $v := v^i e_i|_a$, where $v^i = w(x^i) \in \mathbb{R}$. We want to show that $w = \Phi(v) = D_v|_a$. To this end, let $f \in C^\infty(\mathbb{R}^n)$. By Taylor's theorem we can write

$$f(x) = f(a) + \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a)(x^i - a^i) + \sum_{i,j=1}^n (x^i - a^i)(x^j - a^j) \int_0^1 (1-t) \frac{\partial^2 f}{\partial x^i \partial x^j}(a + t(x-a))dt$$

  Note that each term in the above sum is a product of two smooth functions of x that vanish at $x = a$: one is $(x^i - a^i)$ and the other is $(x^i - a^i)$(integral). By Lemma 5.2 (b) the derivation w annihilates the entire sum. Thus,

$$\begin{aligned}
wf &= w(f(a)) + \sum_{i=1}^n w\left(\frac{\partial f}{\partial x^i}(a)(x^i - a^i)\right) \\
&= \sum_{i=1}^n \frac{\partial f}{\partial x^i}(a)(\underbrace{w(x^i)}_{v^i} - \underbrace{w(a^i)}_{=0}) \\
&= \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i}(a) \\
&= D_v|_a(f)
\end{aligned}$$

(c) We have already used above that if $v = v^i e_i|_a$ (in terms of the standard basis), then $D_v|_a(f) =$

$v^i \frac{\partial f}{\partial x^i}(a)$ by the chain rule. In particular, if $v = e_i|_a$, then $D_{e_i}|_a(f) = \frac{\partial f}{\partial x^i}(a)$ (the i-th derivation defined above), i.e $\frac{\partial}{\partial x^i}\Big|_a = D_{e_i}|_a$. Hence, (c) follows from (b).

□

5.2 Tangent Space

Definition 5.4. Let M be a smooth manifold and let $p \in M$. A map $v : C^\infty(M) \to \mathbb{R}$ is called a derivation at p if it is $\mathbb{R}$-linear and satisfies the product rule:

$$v(fg) = f(p)v(g) + g(p)v(f), \quad \forall f, g \in C^\infty(M).$$

We denote by T_pM the set of all derivations of $C^\infty(M)$ at p. Clearly, T_pM is an $\mathbb{R}$-vector space, called the tangent space to M at $p \in M$. An element of T_pM is called a tangent vector at p.

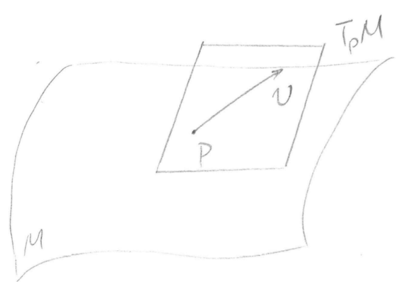
Lemma 5.5. Let M be a smooth manifold, $p \in M$, $v \in T_pM$ and $f, g \in C^\infty(M)$.

(a) If f is constant, then $vf = 0$.

(b) If $f(p) = g(p) = 0$, then $v(fg) = 0$.

Proof. Exercise! (cf. Lemma 5.2) □

With the motivation of geometric tangent vectors in $\mathbb{R}^n$ in mind, we visualize tangent vectors to M as "arrows" that are tangent to M and whose base points are attached to M at the given point.



For alternative descriptions of tangent vectors to M, see Exercise Sheet 4/Exercise Sheet 1.

Definition 5.6. If $F : M \to N$ is a smooth map, then for each $p \in M$ we define a map $dF_p : T_pM \to T_{F(p)}N$, called the differential of F at p, as follows. Given $v \in T_pM$, we let $dF_p(v)$ be the derivation at $F(p)$ that acts on $f \in C^\infty(N)$ by

$$dF_p(v)(f) = v(f \circ F).$$

The operator $dF_p(v) : C^\infty(N) \to \mathbb{R}$ is a derivation at $F(p)$: it is $\mathbb{R}$-linear, since v is so, and satisfies the product rule

$$\begin{aligned} dF_p(v)(fg) &= v((fg)\circ F) = v((f\circ F)(g\circ F)) \\ &= (f\circ F)(p)\, v(g\circ F) + (g\circ F)(p)\, v(f\circ F) \\ &= f(F(p))\, dF_p(v)(g) + g(F(p))\, dF_p(v)(f). \end{aligned}$$

Proposition 5.7. Let $F : M \to N$ and $G : N \to P$ be smooth maps and let $p \in M$.

(a) $dF_p : T_pM \to T_{F(p)}N$ in an $\mathbb{R}$-linear map.

(b) $d(G\circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{(G\circ F)(p)}P$.

(c) $d(\mathrm{Id}_M)_p = \mathrm{Id}_{T_pM} : T_pM \to T_pM$.

(d) If F is a diffeomorphism, then $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism, and it holds that $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.

Proof. Exercise Sheet 4. □

Our first important application of the differential will be to use coordinate charts to relate the tangent space to a point on a manifold with the Euclidean tangent space. But there is an important technical issue that we must address first. While the tangent space is defined in terms of smooth functions on the whole manifold, coordinate charts are in general defined only on open subsets. The key point, expressed in the next proposition, is that tangent vectors act locally.

Proposition 5.8. Let M be a smooth manifold, $p \in M$ and $v \in T_pM$. If $f, g \in C^\infty(M)$ agree on some neighborhood of p, then $vf = vg$.

Proof. Set $h := f - g$ and observe that h is a smooth function on M that vanishes in a neighborhood of p. By Proposition 4.4 there exists a smooth bump function ψ for supp h supported in $M\backslash\{p\}$ (open in M contains supp h, since $h(p) = 0$). Since $\psi \equiv 1$ where h is non-zero, the product ψh is identically equal to h. Since $h(p) = \psi(p) = 0$, by Lemma 5.5(b) we obtain $v(h) = v(\psi h) = 0$, so $v(f) = v(g)$ by linearity. □

By the proposition above, we can identify the tangent space to an open submanifold with the tangent space to the whole manifold.

Proposition 5.9. Let M be a smooth manifold, let $U \subseteq M$ be an open subset, and let $\iota : U \hookrightarrow M$ be the inclusion map. For every point $p \in U$, the differential $d\iota_p : T_pU \to T_pM$ is an isomorphism.

Proof. We first prove injectivity. Let $v \in T_pU$ such that $d\iota_p(v) = 0 \in T_pM$. Let V be a neighborhood of p such that $\bar{V} \subseteq U$. If $f \in C^\infty(U)$ is arbitrary, then by Lemma 4.5, there exists $\tilde{f} \in C^\infty(M)$ such that $\tilde{f}|_{\bar{V}} = f$. Since then f and $\tilde{f}|_U$ are smooth functions that agree in a neighborhood of p, Proposition 5.8 implies

$$vf = v(\tilde{f}|_U) = v(\tilde{f}\circ\iota) = d\iota_p(v)(\tilde{f}) = 0.$$

Hence, $v = 0 \in T_pU$, so $d\iota_p$ is injective.

We now prove surjectivity. Let $w \in T_pM$. Define

$$\begin{aligned} v : C^\infty(U) &\to \mathbb{R} \\ f &\mapsto w\tilde{f}, \end{aligned}$$

where $\tilde{f}$ is any smooth function on M that agrees with f on $\bar{V}$ (see Lemma 4.5). By Proposition 5.8, vf is independent of the choice of $\tilde{f}$, so v is well-defined, and it is easy to check that it is a derivation of $C^\infty(U)$ at p. For any $g \in C^\infty(M)$, we have

$$d\iota_p(v)(g) = v(g \circ \iota) = w(\tilde{g} \circ \iota) = wg,$$

where the last equality follows from the fact that $g \circ \iota$, $\tilde{g} \circ \iota$ and g all agree on V. □

Given an open subset $U \subseteq M$, the isomorphism $d\iota_p$ between T_pU and T_pM is canonically defined, independent of any choices. From now on, we identify T_pU with T_pM for any $p \in U$. This identification just amounts to the observation that $d\iota_p(v)$ is the same derivation as v, thought of as acting on functions on the bigger manifold M instead of on functions on U. Since the action of a derivation on a function depends only on the values of the function in an arbitrarily small neighborhood, this is a harmless identification. In particular, this means that any tangent vector $v \in T_pM$ can be unambiguously applied to functions defined only in a neighborhood of p, not necessarily on all of M.

Proposition 5.10. If M is a smooth n-manifold, then for each $p \in M$, the tangent space T_pM is an n-dimensional $\mathbb{R}$-vector space.

Proof. Fix $p \in M$ and let (U, φ) be a smooth coordinate chart containing p. Since $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ is a diffeomorphism, $d\varphi_p : T_pU \to T_{\varphi(p)}\hat{U}$ is an isomorphism by Proposition 5.7(d). Since Proposition 5.9 guarantees that $T_pU \cong T_pM$ and $T_{\varphi(p)}\hat{U} \cong T_p\hat{U}$, it follows from Proposition 5.3(c) that $\dim T_pM = \dim T_{\varphi(p)}\mathbb{R}^n = n$. □

→ The tangent space to a vector space: Exercise Sheet 4

→ The tangent space to a product manifold: Exercise Sheet 4

Next, we will show how to do computations with tangent vectors and differentials in local coordinates. Let M be a smooth manifold and let (U, φ) be a smooth coordinate chart on M. Then φ is a diffeomorphism from U to an open subset $\hat{U} \subseteq \mathbb{R}^n$. By Propositions 5.7(d) and 5.9, we infer that $d\varphi_p : T_p\mathbf{M} \to T_{\varphi(p)}\mathbb{R}^n$ is an isomorphism (for each $p \in U$). By Prop. 5.3(c) the derivations

$$\left.\frac{\partial}{\partial x^1}\right|_{\varphi(p)}, \ldots, \left.\frac{\partial}{\partial x^n}\right|_{\varphi(p)}$$

form a basis of $T_{\varphi(p)}\mathbb{R}^n$. Therefore, the preimages of these vectors under the isomorphism $d\varphi_p$, denoted by $\frac{\partial}{\partial x^i}|_p$ and characterized by

$$\left.\frac{\partial}{\partial x^i}\right|_p := (d\varphi_p)^{-1}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right) = d(\varphi^{-1})_{\varphi(p)}\left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\right),$$

form a basis of T_pM. Unwinding the definitions, we see that $\frac{\partial}{\partial x^i}|_p$ acts on a function $f \in C^\infty(U)$ by

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)}\left(f \circ \varphi^{-1}\right) = \frac{\partial \hat{f}}{\partial x^i}(\hat{p}),$$

where $\hat{f} = f \circ \varphi^{-1}$ is the coordinate representation of f and $\hat{p} = (p^1, \ldots, p^n) = \varphi(p)$ is the coordinate representation of p. In other words, $\frac{\partial}{\partial x^i}|_p$ is the derivation that takes the i-th partial derivative of (the coordinate representation of) f at (the coordinate representation of) p. The vectors $\frac{\partial}{\partial x^i}|_p$ are called the coordinate vectors at p associated with the given coordinate system.

In summary, if M is a smooth N-manifold and if $p \in M$, then T_pM is an N-dimensional $\mathbb{R}$-vector space, and for any smooth chart $(U, (x^i))$ containing p, the coordinate vectors $\left\{\frac{\partial}{\partial x^i}|_p\right\}_{i=1}^n$ form a basis for T_pM. Thus, a tangent vector $v \in T_pM$ can be written uniquely as a linear combination $v = v^i \frac{\partial}{\partial x^i}|_p$. The ordered basis $\left(\frac{\partial}{\partial x^i}|_p\right)$ is called a coordinate basis for T_pM, and the numbers (v^i) are called the components of v with respect to the coordinate basis. If v is known, then its components can be easily computed from its action on the coordinate functions. For each j, the components of v are given by $v^j = v(x^j)$ (where we think of x^j as a smooth real-valued function on U), because

$$v(x^j) = \left(v^i \left.\frac{\partial}{\partial x^i}\right|_p\right)(x^j) = v^i \frac{\partial x^j}{\partial x^i}(p) = v^j.$$

We now explore how differentials look in coordinates. We begin by considering the case of a smooth map $F : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^m$. For any $p \in U$, we will determine the matrix of $dF_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ in terms of the standard coordinate bases. Denoting by $(x^1, \ldots, x^n)$ the coordinates in the domain (and $y^1, \ldots, y^m$ in the codomain), we use the chain rule to compute the action of dF_p on a typical basis vector as follows:

$$dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) f = \left.\frac{\partial}{\partial x^i}\right|_p (f \circ F) = \frac{\partial f}{\partial y^j}(F(p))\frac{\partial F^j}{\partial x^i}(p) = \left(\frac{\partial F^j}{\partial x^i}(p)\left.\frac{\partial}{\partial y^j}\right|_{F(p)}\right) f.$$

Thus,

$$dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \frac{\partial F^j}{\partial x^i}(p)\left.\frac{\partial}{\partial y^j}\right|_{F(p)}.$$

In other words, the matrix of dF_p in terms of the coordinate bases is

$$\begin{pmatrix} \frac{\partial F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^n}(p) \\ \vdots & & \vdots \\ \frac{\partial F^m}{\partial x^1}(p) & \cdots & \frac{\partial F^m}{\partial x^n}(p) \end{pmatrix}$$

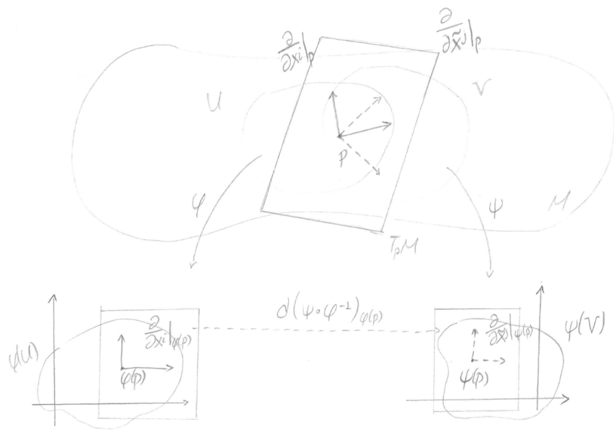
that is, the Jacobian matrix of F at p, which is the matrix representation of the total derivative $DF(p) : \mathbb{R}^n \to \mathbb{R}^m$. Therefore, in this case, $dF_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ corresponds to the total derivative $DF(p) : \mathbb{R}^n \to \mathbb{R}^m$, under the usual identification of Euclidean spaces with their tangent spaces.

We now consider the more general case of a smooth map $F : M \to N$ between smooth manifolds. Choosing smooth coordinate charts (U, φ) for M containing p and (V, ψ) for N containing $F(p)$, we obtain the coordinate representation $\hat{F} = \psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \psi(V)$, and we also denote by $\hat{p} = \varphi(p)$ the coordinate representation of p. By the computation above, $d\hat{F}_p$ is represented with respect to the standard coordinate bases by the Jacobian matrix of $\hat{F}$ at $\hat{p}$. Using the fact that $F \circ \varphi^{-1} = \psi^{-1} \circ \hat{F}$, we compute

$$\begin{aligned} dF_p\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) &= dF_p\left(d(\varphi^{-1})_{\hat{p}}\left(\left.\frac{\partial}{\partial x^i}\right|_{\hat{p}}\right)\right) \\ &= d(F \circ \varphi^{-1})_{\hat{p}}\left(\left.\frac{\partial}{\partial x^i}\right|_{\hat{p}}\right) \\ &= d(\psi^{-1})_{\hat{F}(\hat{p}}\left(d\hat{F}_{\hat{p}}\left(\left.\frac{\partial}{\partial x^i}\right|_{\hat{p}}\right)\right) \\ &= d(\psi^{-1})_{F(p)}\left(\frac{\partial \hat{F}^j}{\partial x^i}(\hat{p})\left.\frac{\partial}{\partial y^j}\right|_{\hat{F}(\hat{p})}\right) \\ &= \frac{\partial \hat{F}^j}{\partial x^i}(\hat{p})\left.\frac{\partial}{\partial y^j}\right|_{F(p)} \end{aligned}$$

Thus, dF_p is represented in coordinate bases by the Jacobian matrix of (the coordinate representation $\hat{F}$ of) F. (In fact, the definition of the differential was cooked up precisely in order to give a coordinate-independent meaning to the Jacobian matrix.)

Finally, suppose that $(U, \varphi = (x^i))$ and $(V, \psi = (\tilde{x}^i))$ are two smooth charts on M, and that $p \in U$. Any tangent vector at p can be represented with respect to either coordinate basis $\left(\frac{\partial}{\partial x^i}|_p\right)$ or $\left(\frac{\partial}{\partial \tilde{x}^i}|_p\right)$. How are the two representations related?



In this situation, it is customary to write the transition map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ in the following shorthand notation:

$$\psi \circ \varphi^{-1}(x) = \left(\tilde{x}^1(x), \ldots, \tilde{x}^n(x)\right).$$

Here we are indulging in a typical abuse of notation: in the expression $\tilde{x}^i(x)$, we think of $\tilde{x}^i$ as a coordinate function (whose domain is an open subset of M, identified with an open subset of $\mathbb{R}^n$), but we think of x as representing a point (in this case, in $\varphi(U \cap V)$). We have

$$d(\psi \circ \varphi^{-1})_{\varphi(p)} \left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} \right) = \frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p)) \left.\frac{\partial}{\partial \tilde{x}^j}\right|_{\psi(p)}.$$

Using the definition of coordinate vectors, we obtain

$$\begin{aligned} \left.\frac{\partial}{\partial x^i}\right|_p &= d(\varphi^{-1})_{\varphi(p)} \left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} \right) \\ &= d(\psi^{-1})_{\psi(p)} \circ d(\psi \circ \varphi^{-1})_{\varphi(p)} \left(\left.\frac{\partial}{\partial x^i}\right|_{\varphi(p)} \right) \\ &= d(\psi^{-1})_{\psi(p)} \left(\frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p)) \left.\frac{\partial}{\partial \tilde{x}^j}\right|_{\psi(p)} \right) \\ &= \frac{\partial \tilde{x}^j}{\partial x^i}(\varphi(p)) \left.\frac{\partial}{\partial \tilde{x}^j}\right|_p. \end{aligned}$$

(This formula looks like the chain rule for partial derivatives in $\mathbb{R}^n$.) Applying this to the components of a vector

$$v = v^i \left.\frac{\partial}{\partial x^i}\right|_p = \tilde{v}^j \left.\frac{\partial}{\partial \tilde{x}^j}\right|_p,$$

we find that the components of v transform by the rule

$$\tilde{v}^j = \frac{\partial \tilde{x}^j}{\partial x^i}(\hat{p}) v^i.$$

5.3 The Tangent Bundle

Definition 5.11. Let M be a smooth manifold. The tangent bundle of M is denoted by TM and is defined as the disjoint union of the tangent spaces at all points of M:

$$TM = \bigsqcup_{p \in M} T_pM.$$

We usually write an element of this disjoint union as an ordered pair (p, v) with $p \in M$ and $v \in T_pM$ (we sometimes write v_p for (p, v)). The tangent bundle comes equipped with a natural projection map $\pi : TM \to M$, which sends each vector in T_pM to the point p at which it is tangent: $(p, v) \mapsto p$.

For example, when $M = \mathbb{R}^n$, using Proposition 5.3, we see that

$$T(\mathbb{R}^n) = \bigsqcup_{p \in \mathbb{R}^n} T_p\mathbb{R}^n \cong \bigsqcup_{p \in \mathbb{R}^n} \mathbb{R}^n_p = \bigsqcup_{p \in \mathbb{R}^n} \{p\} \times \mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n.$$

An element of this Cartesian product can be thought of as representing either the geometric tangent vector v_p, or the derivation $D_v|_p$ defined in Proposition 5.3. In general, however, the tangent bundle of a smooth manifold cannot be identified in a natural way with a Cartesian product, because there is no canonical way to identify tangent spaces at distinct points with each other. The next proposition shows that the tangent bundle of a smooth manifold can be considered as a smooth manifold in its own right. For its proof, we need Lemma 2.9 (smooth manifold chart lemma).

Proposition 5.12. For any smooth N-manifold M, the tangent bundle TM has a natural topology and smooth structure that make it into a smooth $(2n)$-manifold. With respect to this structure, the projection map $\pi : TM \to M$ is smooth.

Proof. We begin by defining the maps that will become our smooth charts. Given any smooth chart (U, φ) for M, observe that $\pi^{-1}(U)$ is the set of all tangent vectors to M at all points of U. Denote by $(x^1, \ldots, x^n)$ the coordinate functions of φ, and define a map

$$\tilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$$

$$\tilde{\varphi}\left(v^i \frac{\partial}{\partial x^i}\bigg|_p \right) = \left(x^1(p), \ldots, x^n(p), v^1, \ldots, v^n \right).$$

Its image is the set $\varphi(U) \times \mathbb{R}^n$, which is an open subset of $\mathbb{R}^{2n}$. It is a bijection onto its image, because its inverse can be explicitly written as

$$\tilde{\varphi}^{-1}(x^1, \ldots, x^n, v^1, \ldots, v^n) = v^i \frac{\partial}{\partial x^i}\bigg|_{\varphi^{-1}(x)}.$$

Now, suppose we are given two smooth charts (U, φ) and (V, ψ) for M, and consider the corresponding "charts" $(\pi^{-1}(U), \tilde{\varphi})$ and $(\pi^{-1}(V), \tilde{\psi})$ for TM. The sets

$$\tilde{\varphi}\left(\pi^{-1}(U) \cap \pi^{-1}(V)\right) = \varphi(U \cap V) \times \mathbb{R}^n$$

and

$$\tilde{\psi}\left(\pi^{-1}(U) \cap \pi^{-1}(V)\right) = \psi(U \cap V) \times \mathbb{R}^n$$

are open in $\mathbb{R}^{2n}$, and the transition map $\tilde{\psi} \circ \tilde{\varphi}^{-1} : \varphi(U \cap V) \times \mathbb{R}^n \to \psi(U \cap V) \times \mathbb{R}^n$ can be written explicitly as

$$\begin{aligned} \tilde{\psi} \circ \tilde{\varphi}^{-1}(x^1, \dots, x^n, v^1, \dots, v^n) &= \tilde{\psi}\left(v^i \frac{\partial}{\partial x^i}\bigg|_{\varphi^{-1}(x)} \right) \\ &= \tilde{\psi}\left(\left(v^i \frac{\partial \tilde{x}^j}{\partial x^i} \right) \frac{\partial}{\partial \tilde{x}^j}\bigg|_{\varphi^{-1}(x)} \right) \\ &= \left(\tilde{x}^1(x), \dots, \tilde{x}^n(x), \frac{\partial \tilde{x}^1}{\partial x^i} v^i, \dots, \frac{\partial \tilde{x}^n}{\partial x^i} v^i \right), \end{aligned}$$

which is clearly smooth.

Choosing a countable cover $\{U_i\}$ of M by smooth coordinate domains, we obtain a countable cover of TM by coordinate domains $\{\pi^{-1}(U_i)\}$ satisfying conditions (i)-(iv) of Lemma 2.9. To check the Hausdorff condition (v), just note that any two points in the same fiber of π lie in one chart, while if (p, v) and (q, w) lie in different fibers, there exist disjoint smooth coordinate domains U and V for M such that $p \in U$ and $q \in V$, and then $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are disjoint coordinate neighborhoods containing (p, v) and (q, w), respectively. This completes the proof of the first part of the statement.

Finally, to check that $\pi : TM \to M$ is smooth, note that with respect to charts (U, φ) for M and $(\pi^{-1}(U), \tilde{\varphi})$ for TM, its coordinate representation $\varphi \circ \pi \circ \tilde{\varphi}^{-1}$ is $\pi(x, v) = x$. □

The coordinates (x^i, v^i) given by $(*_6)$ are called <u>natural coordinates on TM</u>.

→ Global differential: Exercise Sheet 5

→ Basic properties of global differential: Exercise Sheet 5

Proposition 5.13. If M is a smooth N-manifold which can be covered by a single smooth chart (U, φ), then its tangent bundle TM is diffeomorphic to $\varphi(U) \times \mathbb{R}^n$.

Proof. If (U, φ) is a global smooth chart for M, then φ is, in particular, a diffeomorphism from $U = M$ to an open subset $\hat{U} \subset \mathbb{R}^n$. The proof of Proposition 5.12 showed that the natural coordinate chart $\hat{\varphi}$ is a bijection from TM to $U \times \mathbb{R}^n$, and the smooth structure on TM is defined essentially by declaring $\hat{\varphi}$ to be a diffeomorphism. □

Comments: In general, the tangent bundle is not globally diffeomorphic (or even homeomorphic) to a product of the manifold with $\mathbb{R}^n$.

Chapter 6

Maps of constant rank

6.1 Immersions, Submersions and Embeddings

Definition 6.1. Let M and N be smooth manifolds. Given a smooth map $F : M \to N$ and a point $p \in M$, the rank of F at p is defined to be the rank of the linear map $dF_p : T_pM \to T_{F(p)}N$; it is the rank of the Jacobian matrix of F in any smooth chart, or the dimension of the image $\mathrm{Im}(dF_p) \subseteq T_{F(p)}N$. If F has the same rank r at every point, we say that it has constant rank, and we write $\mathrm{rk}\, F = r$.

Recall that the rank of F at each point is bounded above by $\min\{\dim M, \dim N\}$. If the rank of dF_p is equal to this upper bound, then we say that F has full rank at p. If F has full rank everywhere, we say that F has full rank.

Definition 6.2. A smooth map $F : M \to N$ is called:

(a) a smooth immersion if its differential is injective at each point (or equivalently, $\mathrm{rk}\, F = \dim M$),

(b) a smooth submersion if its differential is surjective at each point (or equivalently, $\mathrm{rk}\, F = \dim N$), and

(c) a smooth embedding if it is a smooth immersion that is also a topological embedding, i.e., a homeomorphism onto its image $F(M) \subseteq N$ in the subspace topology.

Comment:

(1) A smooth embedding is a map that is both a topological embedding and a smooth immersion, not just a topological embedding that happens to be smooth; see Exercise 4.5(1).

(2) We will see that smooth immersions and submersions behave locally like injective and surjective linear maps, respectively.

Lemma 6.3. Let $F : M \to N$ be a smooth map. If dF_p is injective (resp. surjective) for some $p \in M$, then p has a neighborhood U such that $F|_U$ is an immersion (resp. submersion).

Proof. If we choose any smooth coordinates for M near p and for N near $F(p)$, either hypothesis means that the Jacobian matrix of F in coordinates has full rank at $p \in M$. By Exercise Sheet 2, Exercise 4, we know that the set of $n \times m$ matrices of full rank is an open subset of $M(n \times m, \mathbb{R})$ (where $m = \dim M$ and $n = \dim N$), so by continuity, the Jacobian of F (in coordinates) has full rank in some neighborhood of $p \in M$. $\square$

Example 6.4. (1) If $\gamma : J \to M$ is a smooth curve in a smooth manifold M, then γ is an immersion if and only if $\gamma'(t) \neq 0$ for all $t \in J$; see Exercise Sheet 4.

(2) If M is a smooth manifold and its tangent bundle TM is given the smooth manifold structure described in Prop. 5.12, then the projection $\pi : TM \to M$ is a smooth submersion. Indeed, we saw that with respect to any smooth local coordinates (x^i) on an open subset $U \subseteq M$ and the corresponding natural coordinates (x^i, v^i) on $\pi^{-1}(U) \subseteq TM$, the coordinate representation of π is $\hat{\pi}(x, v) = x$, and thus

$$J_{\hat{\pi}} = \begin{pmatrix} I_{\dim M} & 0 \\ 0 & 0 \end{pmatrix}$$

.

(3) If M is a smooth manifold and $U \subseteq M$ is an open subset, then the inclusion map $U \hookrightarrow M$ is a smooth embedding.

$\to$ For further examples, see Exercise Sheet 6 and 7.

To understand more fully what it means for a map to be a smooth embedding, it is useful to bear in mind some examples of injective smooth maps that are not smooth embeddings. The next three examples illustrate three rather different ways in which this can happen.

Example 6.5. (1) The map

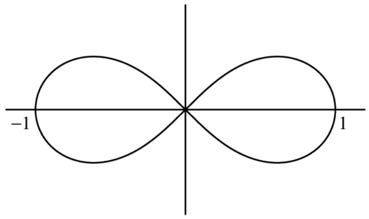
$$\gamma : \mathbb{R} \to \mathbb{R}^2, \quad t \mapsto (t^3, 0)$$

is a smooth map and a topological embedding, but it is not a smooth embedding, because $\gamma'(0) = 0$.

(2) (Lemniscate): Consider the curve

$$\beta : (-\pi, \pi) \to \mathbb{R}^2, \quad t \mapsto (\sin 2t, \sin t)$$

Its image is a set that looks like a figure-eight in the plane (it is the locus of points $(x, y) \in \mathbb{R}^2$ such that $x^2 = 4y^2(1 - y^2)$, as one can easily check). We compute



$$\|\beta(t)\|^2 = \sin^2(t)\,(2\cos 2t + 1)\,, \quad t \in (-\pi, \pi)$$
$$\|\beta'(t)\|^2 = \|\,(2\cos 2t, \cos t)\,\|^2 = 4\cos^2 2t + \cos^2 t, \quad t \in (-\pi, \pi)$$

and hence β is an injective smooth immersion, because $\beta(t_1) = \beta(t_2) \Rightarrow t_1 = t_2$ and $\beta'(t) \neq 0, \forall t \in (-\pi, \pi)$. However, β is not a topological embedding, because its image is compact in the subspace topology, while its domain is not.

(3) Let $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subseteq \mathbb{C}^2$ denote the torus, and let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. The map

$$\gamma : \mathbb{R} \to \mathbb{T}^2, \quad t \mapsto (e^{2\pi i t}, e^{2\pi i \alpha t})$$

is a smooth immersion, because $\gamma'(t)$ never vanishes. It is also injective, because

$$\gamma(t_1) = \gamma(t_2) \Rightarrow t_1 - t_2,\ \alpha t_1 - \alpha t_2 \in \mathbb{Z} \Rightarrow t_1 = t_2.$$

However, γ is not a topological embedding. Indeed, using Dirichlet's approximation theorem [Lee, Lemma 4.21], one can show that $\gamma(0)$ is a limit point of $\gamma(\mathbb{Z}) = \{\gamma(n) \mid n \in \mathbb{Z}\}$, while $\mathbb{Z}$ has no limit point in $\mathbb{R}$. (It can also be shown that $\gamma(\mathbb{R})$ is dense in $\mathbb{T}^2$.)

The following proposition gives a few simple sufficient criteria for an injective immersion to be an embedding.

Proposition 6.6. Let $F : M \to N$ be an injective smooth immersion. If any of the following holds, then F is a smooth embedding:

(a) F is an open or closed map.

(b) F is a proper map (i.e., for every compact subset $K \subset N$, the preimage $F^{-1}(K) \subseteq M$ is compact).

(c) M is compact.

(d) $\dim M = \dim N$.

Proof. We establish some elements from topology.

Claim 1: Let $F : X \to Y$ be a continuous map between topological spaces that is either open or closed. If F is injective, then it is a topological embedding.

- Proof: Assume that F is open and injective. Then $F : X \to F(X)$ is bijective, so $F^{-1} : F(X) \to X$ exists. If $U \subseteq X$ is open, then $(F^{-1})^{-1}(U) = F(U)$ is open in Y by hypothesis and therefore is also open in $F(X)$ by definition of the subspace topology on

$F(X)$. Hence, F^{-1} is continuous, so that F is a topological embedding. The proof of the assertion is similar when F is closed and injective.

Claim 2: (Closed map lemma) Let X be a compact space, Y be a Hausdorff space, and $F : X \to Y$ be a continuous map. Then F is a closed map.

- Proof: Let $K \subseteq X$ be a closed subset. Since X is compact, K is also compact, and since F is continuous, $F(K)$ is also compact. Since Y is Hausdorff, $F(K) \subseteq Y$ is a closed subset. Thus, F is a closed map.

Claim 3: Let X be a topological space, and let Y be a locally compact Hausdorff space. Then every proper continuous map $F : X \to Y$ is closed.

- Proof: Let $K \subseteq X$ be a closed subset. To show that $F(K) \subseteq Y$ is closed, we need to show that its complement is open. Since $Y \setminus F(K)$ is locally compact, Y has an open neighborhood V with compact closure in Y, $F(K)$. Set $U := V \setminus F(E)$, where $E = K \cap F^{-1}(V)$, and note that E is a compact set. Since F is continuous, $F(E)$ is also compact, and since Y is Hausdorff, $F(E)$ is a closed subset of Y. Set $U := V \setminus F(E)$, and observe that U is an open neighborhood of y, which is disjoint from $F(K)$. Hence, $Y \setminus F(K)$ is open, which implies that $F(K)$ is closed.

(a) By Claim 1, F is a topological embedding, and by assumption, F is a smooth immersion, so it is a smooth embedding.

(b) By assumption and by Claim 3, F is a closed map, so it is a smooth embedding by (a).

(c) By assumption and by Claim 2, F is a closed map, so it is a smooth embedding by (a).

(d) By assumption and by Exercise Sheet 6, Exercise 5(b), F is a local diffeomorphism (see Exercise Sheet 6), and thus an open map by Exercise Sheet 6, Exercise 4(c), so it is a smooth embedding by (a).

□

Theorem (Inverse function theorem for manifolds). Let $F : M \to N$ be a smooth map. If $p \in M$ is a point such that the differential dF_p at F is invertible, then there exist connected neighborhoods U_0 of p in M and V_0 of $F(p)$ in N such that $F|_{U_0} : U_0 \to V_0$ is a diffeomorphism.

→ Proof of IFT: Exercise Sheet 6, Exercise 3

→ Local diffeomorphisms are discussed in Exercise Sheet 6

→ ∃ smooth embeddings which are neither open nor closed maps.

The most important fact about maps of constant rank is the following consequence of the inverse function theorem, which says that a smooth map of constant rank can be placed locally into a particularly simple canonical form by a change of coordinates. (This is a non-linear version of the canonical form theorem for linear maps; see [Lee [1], Theorem B.20]).

6.2 The Rank Theorem

Theorem 6.7 (Rank Theorem)**.** Let M and N be smooth manifolds of dimension M and N, respectively, and let $F : M \to N$ be a smooth map of constant rank r. For each $p \in M$, there exist smooth charts (U, φ) for M centered at p and (V, ψ) for N centered at $F(p)$ such that $F(U) \subseteq V$, in which F has a coordinate representation of the form:

$$\hat{F}(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m) = (x^1, \ldots, x^r, 0, \ldots, 0).$$

In particular, if F is a smooth submersion, this becomes:

$$\hat{F}(x^1, \ldots, x^n, x^{n+1}, \ldots, x^m) = (x^1, \ldots, x^n),$$

while if F is a smooth immersion, this becomes:

$$\hat{F}(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0).$$

Proof. Since the theorem is local, after choosing smooth coordinates we can replace M and N by open subsets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$. The fact that $DF(p)$ has rank r implies that its matrix has some $r \times r$ submatrix with non-zero determinant. By reordering the coordinates, w.m.a.t., it is the upper-left submatrix $\left(\frac{\partial F^i}{\partial x^j}\right)$, for $i, j \in \{1, \ldots, r\}$. We relabel the standard coordinates as:

$$(x, y) = (x^1, \ldots, x^r, y^1, \ldots, y^{m-r}) \text{ in } \mathbb{R}^m,$$

and

$$(v, w) = (v^1, \ldots, v^r, w^1, \ldots, w^{n-r}) \text{ in } \mathbb{R}^n.$$

By initial translation of the coordinates without loss of generality, we may assume that let $p = (0, 0)$ and $F(p) = (0, 0)$. If we write $F(x, y) = (Q(x, y), R(x, y))$ for some smooth maps $Q : U \to \mathbb{R}^r$ and $R : U \to \mathbb{R}^{n-r}$, then our hypothesis is that $\partial Q^i / \partial x^j$ is non-singular at $(0, 0)$.

Define

$$\varphi : U \to \mathbb{R}^m, \quad \varphi(x, y) = (Q(x, y), y)$$

and observe that its total derivative at $(0,0)$ is:

$$D\varphi(0,0) = \begin{pmatrix} \dfrac{\partial Q^i}{\partial x^j}(0,0) & \dfrac{\partial Q^i}{\partial y^j}(0,0) \\ 0 & \delta^i_j \end{pmatrix}, \qquad (\delta^i_j \text{ Kronecker delta})$$

which is non-singular by virtue of the hypothesis. Therefore, by the inverse function theorem, there are connected neighborhoods U_0 of $(0,0)$ and $\tilde{U}_0$ of $\varphi(0,0) = (0,0)$ such that $\varphi|_{U_0} : U_0 \to \tilde{U}_0$ is a diffeomorphism. By shrinking U_0 and $\tilde{U}_0$ if necessary, we may assume that $\tilde{U}_0$ is an open cube. Writing the inverse map as:

$$\varphi^{-1}(x,y) = (A(x,y), B(x,y)),$$

for some smooth maps $A : \tilde{U}_0 \to \mathbb{R}^r$ and $B : \tilde{U}_0 \to \mathbb{R}^{m-r}$, we compute:

$$(x,y) = \varphi(A(x,y), B(x,y)) = (Q(A(x,y), B(x,y)), B(x,y)),$$

Comparing the y-components shows that $B(x,y) = y$, and therefore φ^{-1} has the form:

$$\varphi^{-1}(x,y) = (A(x,y), y).$$

On the other hand, $\varphi \circ \varphi^{-1} = \mathrm{Id}$ implies $Q(A(x,y),y) = x$, and therefore $F \circ \varphi^{-1}$ has the form:

$$(F \circ \varphi^{-1})(x,y) = (x, \tilde{R}(x,y)),$$

where $\tilde{R} : \tilde{U}_0 \to \mathbb{R}^{n-r}$ is defined by $\tilde{R}(x,y) = R(A(x,y),y)$. The Jacobian matrix of $F \circ \varphi^{-1}$ at an arbitrary point $(x,y) \in \tilde{U}_0$ is

$$D(F \circ \varphi^{-1})(x,y) = \begin{pmatrix} \delta^i_j & 0 \\ \dfrac{\partial \tilde{R}^i}{\partial x^j}(x,y) & \dfrac{\partial \tilde{R}^i}{\partial y^j}(x,y) \end{pmatrix}.$$

Since composing with a diffeomorphism does not change the rank of a map, the above matrix has rank r everywhere in $\tilde{U}_0$. The first r columns are obviously linearly independent, so the rank can be r only if $\frac{\partial \tilde{R}^i}{\partial y^j}$ vanish identically on $\tilde{U}_0$, which implies that $\tilde{R}$ is actually independent of $(y^1, \ldots, y^{m-r})$. (This is one reason we arranged for $\tilde{U}_0$ to be a cube). Thus, if we let $S(x) = \tilde{R}(x,0)$, then we have

$$(F \circ \varphi^{-1})(x,y) = (x, S(x)).$$

To complete the proof, we need to define an appropriate smooth chart in some neighborhood of $(0,0) \in V$. Consider the open subset

$$V_0 = \{(v,w) \in V \mid (v,0) \in \tilde{U}_0\} \subseteq V,$$

and note that V_0 is an open neighborhood of $(0,0)$. Since $\tilde{U}_0 \ni (0,0) = \varphi(0,0)$ is a cube and

$(F\circ\varphi^{-1})(x,y) = (x,S(x))$, it follows that $(F\circ\varphi^{-1})(\tilde{U}_0) \subseteq V_0$ (because $(u,w) \in \tilde{U}_0 \implies (F\circ\varphi^{-1})(u,w) = (v,S(v)) \in V$ and $(v,0) \in \tilde{U}_0$ by construction of $\tilde{U}_0$), so $F(U_0) \subseteq V_0$. Define

$$\psi : V_0 \to \mathbb{R}^n, \quad \psi(v,w) = (v, w - S(v)).$$

This is an open map and a diffeomorphism because its inverse is given explicitly by $\psi^{-1}(s,t) = (s, t+S(s))$. Thus, (V_0,ψ) is a smooth chart. It follows from $(F\circ\varphi^{-1})(x,y) = (x,S(x))$ that:

$$(\psi\circ F\circ\varphi^{-1})(x,y) = \psi(x,S(x)) = (x, S(x) - S(x)) = (x,0),$$

which was to be proved. □

The next corollary can be viewed as a more invariant statement of the rank theorem. It says that maps of constant rank are precisely the ones whose local behavior is the same as that of their differentials.

Corollary 6.8. Let $F : M \to N$ be a smooth map. Assume that M is connected. Then, the following are equivalent

(a) For each $p \in M$, there exist smooth charts containing p and $F(p)$ in which the coordinate representation of F is linear.

(b) F has constant rank.

Proof. The proof is a direct application of the constant rank theorem.

- (b)⇒(a): Follows from the rank theorem.
- (a)⇒(b): Since every linear map has constant rank, it follows that the rank of F is constant in a neighborhood of each point, and thus, by connectedness, it is constant on all of M.

□

Theorem 6.9 (Global Rank Theorem)**.** Let $F : M \to N$ be a smooth map of constant rank.

(a) If F is surjective, then it is a smooth submersion.

(b) If F is injective, then it is a smooth immersion.

(c) If F is bijective, then it is a diffeomorphism.

Proof. Assume that $m = \dim M$, $n = \dim N$, and $r = \mathrm{rk} F$.

(a) See [Lee, Thm 4.14(a)].

(b) Assume that F is not a smooth immersion, so that $r < m$. By the rank theorem, for each $p \in M$, we can choose charts around p and $F(p)$ in which F has the coordinate representation

$$\hat{F}(x^1,\dots,x^r,x^{r+1},\dots,x^m) = (x^1,\dots,x^r,0,\dots,0).$$

Thus, $\hat{F}(0,\dots,0,\varepsilon) = (0,\dots,0)$ for any $0 < \varepsilon \ll 1$, which shows that F is not injective, a contradiction.

(c)

$$
\begin{aligned}
F\text{: bijective} \quad &\Longrightarrow \quad F\text{: injective + surjective} \\
&\xRightarrow[\text{(b)}]{\text{(a)}} \quad F\text{: smooth immersion + smooth submersion} \\
&\xRightarrow{\text{ES6E5(a)}} \quad F\text{: local diffeomorphism} \\
&\xRightarrow[F\text{: bijective}]{\text{ES6E4(f)}} \quad F\text{: diffeomorphism}
\end{aligned}
$$

□

6.3 Local and Global Sections

Let $\pi : M \to N$ be a continuous map.

- A section of π is a continuous right inverse for π, i.e., a continuous map $\sigma : N \to M$ such that $\pi \circ \sigma = \mathrm{Id}_N$.
- A local section of π is a continuous map $\sigma : U \to M$ defined on some open subset $U \subseteq N$ and satisfying the analogous relation $\pi \circ \sigma = \mathrm{Id}_U$.

Many of the important properties of smooth submersions follow from the fact that they admit an abundance of smooth local sections, which we prove below.

Theorem 6.10 (Local Section Theorem)**.** Let $\pi : M \to N$ be a smooth map. Then π is a smooth submersion if and only if every point of M is in the image of a smooth local section of π.

Proof. Set $m := \dim M$ and $n := \dim N$.

"$\Rightarrow$": Fix $p \in M$ and set $q = \pi(p)$. By the rank theorem, we can choose smooth coordinates $(x^1,\dots,x^m)$ centered at p and $(y^1,\dots,y^n)$ centered at q, in which π has the coordinate representation:

$$\pi(x^1,\dots,x^n,x^{n+1},\dots,x^m) = (x^1,\dots,x^n).$$

If ε is a sufficiently small positive real number, then the coordinate cube

$$C_\varepsilon = \{x \mid |x^i| < \varepsilon, 1 \le i \le m\}$$

is a neighborhood of p whose image under π is the cube

$$C'_\varepsilon = \{y \mid |y^i| < \varepsilon, 1 \le i \le n\}.$$

The map $\sigma : C'_\varepsilon \to C_\varepsilon$ given by

$$\sigma(x^1, \dots, x^n) = (x^1, \dots, x^n, 0, \dots, 0)$$

is a smooth local section of π satisfying $\sigma(q) = p$.

"$\Leftarrow$": Given $p \in M$, let $\sigma : U \to M$ be a smooth local section of π such that $\sigma(q) = p$, where $q = \pi(\sigma(q)) = \pi(p)$. The equation $\pi \circ \sigma = \mathrm{Id}_U$ implies that $d\pi_p \cdot d\sigma_q = \mathrm{Id}_{T_qN}$ by Proposition 5.7(b), which in turn implies that $d\pi_p$ is surjective. Since $p \in M$ was arbitrary, we conclude that π is a smooth submersion.

$\square$

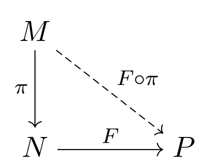
Recall: If X is a topological space, Y is a set, and $\pi : X \to Y$ is a surjective map, then the quotient topology on Y determined by π is defined by declaring a subset $V \subseteq Y$ to be open if $\pi^{-1}(V)$ is open in X. If X and Y are topological spaces, a map $\pi : X \to Y$ is called a quotient map if it is surjective and continuous, and Y has the quotient topology determined by π.

Proposition 6.11. Let $\pi : M \to N$ be a smooth submersion. Then π is an open map, and if it is surjective, then it is a quotient map.

Proof. The second assertion follows from the first one (a surjective, open, continuous map is a quotient map), so we now prove that π is an open map. Let W be an open subset of M and let $q \in \pi(W)$. For any $p \in W$ such that $\pi(p) = q$, by Theorem 6.10 there is a neighborhood U of q on which there exists a smooth local section $\sigma : U \to M$ with $\sigma(q) = p$. For each $y \in \sigma^{-1}(W)$, the fact that $\sigma(y) \in W$ implies that $y = \pi(\sigma(y)) \in \pi(W)$. Thus, $\sigma^{-1}(W)$ is an open neighborhood of q contained in $\pi(W)$, which implies that $\pi(W)$ is open. $\square$

The next three theorems provide important tools that are frequently used when studying submersions and demonstrate that surjective smooth submersions play a role in smooth manifold theory analogous to the role of quotient maps in topology.

Theorem 6.12 (Characteristic Property of Surjective Smooth Submersions)**.** Let $\pi : M \to N$ be a surjective smooth submersion. For any smooth manifold P, a map $F : N \to P$ is smooth if and only if $F \circ \pi : M \to P$ is smooth.



Proof. If F is smooth, then $F \circ \pi$ is also smooth by Proposition 3.6(d). Conversely, assume that $F \circ \pi$ is smooth and let $q \in N$. Since π is surjective, there is $p \in M$ such that $\pi(p) = q$, and then Theorem 4.10 guarantees the existence of a neighborhood U of q in N and a smooth local section $\sigma : U \to M$ at π such that $\sigma(q) = p$. Then $\pi \circ \sigma = \mathrm{Id}_U$ implies

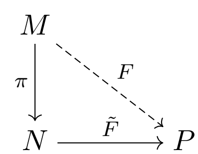
$$F|_U = F|_U \circ \mathrm{Id}_U = F|_U \circ (\pi \circ \sigma) = (F \circ \pi) \circ \sigma,$$

which is a composition of smooth maps. Hence, F is smooth by Proposition 3.6(d)=Exercise Sheet 3, Exercise 3(e) and Exercise 2(a). □

→ Exercise Sheet 7, Exercise 4 explains the sense in which this property is ”characteristic”

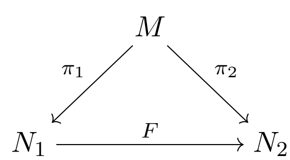
→ Exercise Sheet 7, Exercise 5 shows that the converse of Theorem 6.12 is false.

Theorem 6.13 (Pushing smoothly to the quotient)**.** Let $\pi : M \to N$ be a surjective smooth submersion. If P is a smooth manifold and if $F : M \to P$ is a smooth map that is constant on the fibers of π, then there exists a unique smooth map $\tilde{F} : N \to P$ such that $\tilde{F} \circ \pi = F$.



Proof. By Proposition 6.11, π is a quotient map, and by [Lee [1], Theorem A.30], there exists a unique continuous map $\tilde{F} : N \to P$ such that $\tilde{F} \circ \pi = F$. This map is smooth by Theorem 6.12. □

Theorem 6.14 (Uniqueness of smooth quotients)**.** Let $\pi_1 : M \to N_1$ and $\pi_2 : M \to N_2$ be surjective smooth submersions that are constant on each other's fibers. Then there exists a unique diffeomorphism $F : N_1 \to N_2$ such that $F \circ \pi_1 = \pi_2$.



Proof. Exercise Sheet 7, Exercise 6. □

Chapter 7

Submanifolds

7.1 Embedded Submanifolds

Definition 7.1. Let M be a smooth manifold. An embedded submanifold of M is a subset $S \subseteq M$ that is a topological manifold in the subspace topology, endowed with a smooth structure such that the inclusion map $S \hookrightarrow M$ is a smooth embedding.

If S is an embedded submanifold of M, then the difference $\dim M - \dim S$ is called the codimension of S in M, and the containing manifold M is called the ambient manifold for S.

(The empty set $\emptyset$ is an embedded submanifold of any dimension.)

Proposition 7.2 (Open submanifolds)**.** Let M be a smooth manifold. The embedded submanifolds of codimension 0 in M are exactly the open submanifolds.

Proof. If $U \subseteq M$ is an open submanifold (Exercise 1.8(3)), then we have already seen that U is a smooth manifold of $\dim U = \dim M$ and that the inclusion map $L : U \hookrightarrow M$ is a smooth embedding (Exercise 4.4(3)), so $U = M$ is an embedded submanifold of codimension 0.

Conversely, let $U \subseteq M$ be an embedded submanifold of codimension 0. Then the inclusion $\iota : U \hookrightarrow M$ is a smooth embedding, and thus a local diffeomorphism by Exercise Sheet 6, Exercise 5(b), since $\dim U = \dim M$, so it is an open map by Exercise Sheet 6, Exercise 4(c). Therefore, U is an open subset of M. □

Proposition 7.3 (Images of embeddings as submanifolds)**.** Let $F : N \to M$ be a smooth embedding and set $S := F(N)$. With the subspace topology, S is a topological manifold, and it has a unique smooth structure making it into an embedded submanifold of M with the property that F is a diffeomorphism onto its image.

Proof. If we give S the subspace topology that it inherits from M, then the assumption that F is an embedding means that F can be considered as a homeomorphism from N onto S, and thus S is a topological manifold. We now give S a smooth structure by taking the smooth charts to be those of the form $(F(U), \varphi \circ F^{-1})$, where (U, φ) is a smooth chart for N; note that the smooth compatibility of these charts follows from the smooth compatibility of the corresponding charts for

N. With this smooth structure on S, the map F is a diffeomorphism onto its image (essentially by definition), and this is obviously the only smooth structure with this property. Finally, the inclusion map $L : S \hookrightarrow M$ is equal to the composition of a diffeomorphism followed by a smooth embedding

$$S \xrightarrow{F^{-1}} N \xrightarrow{F} M,$$

so it is a smooth embedding by Exercise Sheet 6, Exercise 1(a)(iii). $\square$

Since every embedded submanifold is the image of a smooth embedding (namely its own inclusion map), Proposition 7.3 shows that embedded submanifolds are exactly the images of smooth embeddings.

Proposition 7.4. (Graphs as submanifolds): Let M be a smooth m-manifold, let N be a smooth n-manifold, let $U \subseteq M$ be an open subset, and let $f : U \to N$ be a smooth map. Then the graph of f,

$$\Gamma(f) := \{(x, y) \in M \times N \mid x \in U, y = f(x)\},$$

is an embedded m-dimensional submanifold of $M \times N$ diffeomorphic to U.

Proof. (Recall Exercise 1.3(1) and 1.8(1)). Consider the map

$$\gamma_f : U \to M \times N, \quad x \mapsto (x, f(x)).$$

It is a smooth map whose image is $\Gamma(f)$. Since the projection $\pi_M : M \times N \to M$ satisfies $\pi_M \circ \gamma_f(x) = \mathrm{Id}_U(x) = x$ for $x \in U$, the composition $d(\pi_M)_{(x,f(x))} \circ d(\gamma_f)_x$ is the identity on T_xM for each $x \in U$. Thus, $d(\gamma_f)_x$ is injective, so γ_f is a smooth immersion. It is also a homeomorphism onto its image, since $\pi_M|_{\Gamma(f)}$ is a continuous inverse for it. Thus, $\Gamma(f)$ is an embedded submanifold of $M \times N$ diffeomorphic to U by Proposition 7.3. $\square$

In particular, if M and N are smooth manifolds, then for each $q \in N$, the subset $M \times \{q\}$, called a slice of the product manifold, is an embedded submanifold of $M \times N$ diffeomorphic to M by Proposition 7.4 and Exercise Sheet 3, Exercise 3(b).

An embedded submanifold $S \subseteq M$ is said to be properly embedded if the inclusion $S \hookrightarrow M$ is a proper map. It will be shown in Exercise Sheet 8, Exercise 1(b) that an embedded submanifold $S \subseteq M$ is properly embedded if and only if S is a closed subset of M. Consequently, every compact embedded submanifold is properly embedded, since compact subset of Hausdorff spaces are closed.

7.2 k-Slice and Level Set Theorems

Definition 7.5. (a) Given an open subset $U \subseteq \mathbb{R}^n$ and $k \in \{0, \dots, n\}$, a k-dimensional slice of U (or simply a k-slice) is any subset of the form

$$S = \{(x^1, \dots, x^k, x^{k+1}, \dots, x^n) \in U \mid x^{k+1} = c^{k+1}, \dots, x^n = c^n\}$$

for some constants $c^{k+1}, \dots, c^n \in \mathbb{R}$ (often taken to be zero). When $k = 0$, then $S \subseteq \{\text{pt}\} \subseteq U$, while when $k = n$, then $S = U$. Note that every k-slice is homeomorphic to an open subset of $\mathbb{R}^k$.

(b) Let M be a smooth manifold and let (U, φ) be a smooth chart for M. If S is a subset of U such that $\varphi(S)$ is a k-slice of $\varphi(U) \subseteq \mathbb{R}^n$, then we say that S is a k-slice of U.

Given a subset $S \subset M$ and $k \in \mathbb{N}$, we say that S satisfies the local k-slice condition if each point of S is contained in the domain of a smooth chart (U, φ) for M such that $S \cap U$ is a single k-slice in U. Any such chart is called a slice chart for S in M, and the corresponding coordinates $(x^1, \dots, x^n)$ are called slice coordinates.

Theorem 7.6 (Local slice criterion for embedded submanifolds)**.** Let M be a smooth n-manifold. If S is an embedded k-dimensional submanifold of M, then S satisfies the local k-slice condition. Conversely, if $S \subseteq M$ is a subset that satisfies the local k-slice condition, then with the subspace topology, S is a topological manifold of dimension k, and it has a smooth structure making it into a k-dimensional embedded submanifold of M.

Proof. "$\Rightarrow$": Since the inclusion map $\iota : S \hookrightarrow M$ is in particular a smooth immersion, by the rank theorem we infer that for any $p \in S$ there are smooth charts (U, φ) for S and (V, ψ) for M, both centered at p, in which the inclusion map $\iota|_U : U \hookrightarrow V$ has the coordinate representation $(x^1, \dots, x^k) \mapsto (x^1, \dots, x^k, 0, \dots, 0)$. Now, choose $0 < \epsilon \ll 1$ so that both U and V contain coordinate balls $U_0 \subseteq U$ and $V_0 \subseteq V$ of radius $\epsilon > 0$ centered at p. It follows that $U_0 \cong \iota(U_0)$ is exactly a single slice in V_0. Since $S \subseteq M$ has the subspace topology, and since U_0 is open in S, there is an open subset $W \subseteq M$ such that $U_0 = W \cap S$. Setting $V_1 = W \cap V_0$, we obtain a smooth chart $(V_1, \psi|_{V_1})$ for M containing p such that $V_1 \cap S = U_0 \cap V_0 = U_0$, which is a single slice of V_1.

"$\Leftarrow$": With the subspace topology, S is Hausdorff and second-countable, because both properties are inherited by subspaces. To show that S is locally Euclidean, we construct an atlas. (The idea of the construction is that if $(x^1, \dots, x^n)$ are slice coordinates for S in M, then we can use $(x^1, \dots, x^k)$ as local coordinates for S.)

Let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be the projection onto the first k-coordinates. Let (U, φ) be a slice chart for S in M, and define $V := U \cap S$, $\hat{V} := (\pi \circ \varphi)(V)$, $\psi := \pi \circ \varphi|_V$. By definition of slice charts, $\varphi(V)$ is the intersection of $\varphi(U)$ with a certain k-slice $A \subseteq \mathbb{R}^n$ defined by setting $x^{k+1} = c^{k+1}, \dots, x^n = c^n$, and thus $\varphi(V)$ is open in A. Since $\pi|_A : A \to \mathbb{R}^k$ is a diffeomorphism, it follows that $\hat{V}$ is open in $\mathbb{R}^k$. Moreover, ψ is a homeomorphism because it has a continuous inverse given by $\varphi^{-1} \circ j|_{\hat{V}}$, where $j : \mathbb{R}^k \to \mathbb{R}^n$ is defined by

$$j(x^1, \ldots, x^k) = (x^1, \ldots, x^k, c^{k+1}, \ldots, c^n).$$

Thus, S is a topological k-manifold, and the inclusion map $i : S \to M$ is clearly a topological embedding.

We now check that the charts constructed above are smoothly compatible. Let (U, φ) and (U', φ') be two slice charts for S in M, and let (V, ψ) and (V', ψ') be the corresponding charts for S. The transition map is given by $\psi' \circ \psi^{-1} = \pi \circ \varphi' \circ \varphi^{-1} \circ j$, which is smooth as a composite of four smooth maps. Hence, the atlas we have constructed is actually a smooth atlas, and it defines a smooth structure on S. In terms of a slice chart (U, φ) for M and the corresponding chart (V, ψ) for S, $i : S \to M$ has a coordinate representation of the form $(x^1, \ldots, x^k) \mapsto (x^1, \ldots, x^k, c^{k+1}, \ldots, c^n)$, so it is a smooth immersion, and we are done by the previous paragraph. □

Notice that the local slice condition for $S \subset M$ is a condition on the subset S only; it does not presuppose any particular topology or smooth structure on S. According to ES8E6, the smooth manifold structure constructed in Theorem 7.6 is the unique one in which S can be considered as a submanifold, so a subset satisfying the local slice condition is an embedded submanifold in only one way.

Definition 7.7. Let $\Phi : M \to N$ be a map. If $c \in N$, then $\Phi^{-1}(c)$ is called a level set of Φ. (In the special case $N = \mathbb{R}^k$ and $c = 0$, the level set $\Phi^{-1}(0)$ is usually called the zero set of Φ.)

Assume now that Φ is a smooth map. A point $p \in M$ is called a regular point of Φ if $d_p\Phi : T_pM \to T_{\Phi(p)}N$ is surjective; otherwise, we say that p is a critical point of Φ. A point $c \in N$ is called a regular value of Φ if every point of the level set $\Phi^{-1}(c)$ is a regular point; otherwise, we say that c is a critical value of Φ. (In particular, if $\Phi^{-1}(c) = \varnothing$, then c is a regular value.) Finally, a level set $\Phi^{-1}(c)$ is called a regular level set if c is a regular value of Φ.

Remark 7.8. Let $\Phi : M \to N$ be a smooth map.

1. If $\dim M < \dim N$, then every point of M is a critical point of Φ.

2. Every point of M is regular if and only if Φ is a smooth submersion.

3. By Lemma 6.3, the set of regular points of Φ is an open subset of M (but may be empty).

Consider the three smooth functions

$$\begin{aligned} \Theta :&\quad \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto x^2 - y \\ \Phi :&\quad \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto x^2 - y^2 \\ \Psi :&\quad \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto x^2 - y^3. \end{aligned}$$

Although the zero set $\Theta^{-1}(0)$ of Θ is an embedded submanifold of $\mathbb{R}^2$, it will be shown in Exercise Sheet 8, Exercise 3(b) and Exercise Sheet 9, Exercise 4(c) that neither the zero set $\Phi^{-1}(0)$ of Φ nor the zero set $\Psi^{-1}(0)$ of Ψ is an embedded submanifold of $\mathbb{R}^2$. Hence, it is fairly easy to find level sets of smooth functions that are not smooth submanifolds. In fact, without further assumptions on the smooth function, the situation is about as bad as could be imagined: according to [Lee [1], Theorem 2.29], every closed subset of M can be expressed as the zero set of a smooth non-negative real-valued function.

Theorem 7.9 (Constant-rank level set theorem)**.** Let $\Phi : M \to N$ be a smooth map of constant rank r. Each level set of Φ is a properly embedded submanifold of codimension r in M.

In particular, if Φ is a smooth submersion, then each level set of Φ is a properly embedded submanifold of M of codimension $r = \dim N$.

Proof. Set $m := \dim M$, $n := \dim N$, and $k := m - r$. Pick $c \in N$ and set $S := \Phi^{-1}(c)$. By the rank theorem, for each $p \in S$ there are smooth charts (U, φ) centered at p and (V, ψ) centered at $c = \Phi(p)$ in which Φ has a coordinate representation of the form

$$\Phi(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m) = (x^1, \ldots, x^r, 0, \ldots, 0),$$

and hence $S \cap U = \Phi^{-1}(0) \cap U$ is the slice

$$\{(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m) \in U \mid x^1 = \cdots = x^r = 0\}.$$

Therefore, S satisfies the local $(k = m - r)$-slice condition, so it is an embedded submanifold of dimension k by Theorem 7.6. It is closed in M by continuity of Φ, so it is properly embedded by Exercise Sheet 8, Exercise 1(b). □

Corollary 7.10 (Regular level set theorem)**.** Every regular level set of a smooth map between smooth manifolds is a properly embedded submanifold whose codimension is equal to the dimension of the codomain.

Proof. Let $\Phi : M \to N$ be a smooth map and let $c \in N$ be a regular value of Φ. By Lemma 7.3, the set

$$U = \{p \in M \mid \operatorname{rk}(d\Phi_p) = \dim N\} \subseteq M$$

is open in M, and contains $\Phi^{-1}(c)$ by assumption. Thus, $\Phi|_U : U \to N$ is a smooth submersion, so $\Phi^{-1}(c)$ is an embedded submanifold of U by Theorem 7.9. It follows now from Proposition 7.2 and Exercise Sheet 8, Exercise 1(a)(iii) that

$$\Phi^{-1}(c) \hookrightarrow U \hookrightarrow M$$

is a smooth embedding, so $\Phi^{-1}(c)$ is an embedded submanifold of M, and it is closed (so properly embedded by Exercise Sheet 8, Exercise 1(b)). □

Not all embedded submanifolds can be expressed as level sets of smooth submersions. However, the next proposition shows that every embedded submanifold is at least locally of this form.

Proposition 7.11. Let S be a subset of a smooth m-manifold M. Then S is an embedded k-submanifold of M if and only if every point of S has a neighborhood U in M such that $U \cap S$ is a level set of a smooth submersion.

Proof. Exercise Sheet 8, Exercise 4. □

If $S \subseteq M$ is an embedded submanifold, a smooth map $\Phi : M \to N$ such that S is a regular level set of Φ is called a defining map for S. (In the special case $N = \mathbb{R}^{m-k}$, it is usually called a defining function for S. For several examples, see ES8 and ES9.) More generally, if $U \subseteq M$ is an open subset and $\Phi : U \to N$ is a smooth map such that $S \cap U$ is a regular level set of Φ, then Φ is called a local defining map (or local defining function) for S. Proposition 7.11 says that every embedded submanifold admits a local defining function in a neighborhood of each of its points.

7.3 Immersed Submanifolds

Definition 7.12. Let M be a smooth manifold. An immersed submanifold of M is a subset $S \subseteq M$ endowed with a topology (not necessarily the subspace topology) with respect to which it is a topological manifold, and a smooth structure with respect to which the inclusion map $S \to M$ is an (injective) smooth immersion. The codimension of S in M is defined as $\dim M - \dim S$.

Observe that every embedded submanifold is an immersed submanifold, but the converse fails in general; see, for instance, Exercise Sheet 8, Exercise 3(b) and Exercise Sheet 9, Exercise 4(b) for a counterexample.

Proposition 7.13 (Images of immersions as submanifolds)**.** Let $F : N \to M$ be an injective smooth immersion. Set $S := F(N)$. Then S has a unique topology and smooth structure such that it is an immersed submanifold of M and such that $F : N \to S$ is a diffeomorphism onto its image.

Proof. We give S a topology by declaring a subset $U \subseteq S$ to be open if and only if $F^{-1}(U) \subseteq N$ is open, and then we give it a smooth structure by taking the smooth charts to be those of the form $(F(U), \varphi \circ F^{-1})$, where (U, φ) is a smooth chart for N. (As in the proof of Proposition 7.3, the smooth compatibility of these charts follows from the smooth compatibility of the corresponding charts for N.) With this topology and smooth structure on S, the map F is a diffeomorphism onto its image, and these are the only topology and smooth structure on S with this property. The inclusion map $\iota : S \to M$ can be written as the composition

$$S \xrightarrow{F^{-1}} N \xrightarrow{F} M,$$

where the first map is a diffeomorphism and the second map is a smooth immersion, so L is also a smooth immersion by Exercise Sheet 6, Exercise 1(a)(ii) and Exercise Sheet 6, Exercise 5(a). □

Example 7.14. The figure-eight (lemniscate) from Exercise 4.5(2) is the image of the injective smooth immersion

$$B : (-\pi, \pi) \to \mathbb{R}^2, \quad t \mapsto (\sin 2t, \sin t)$$

(which is not an embedding), so it is an immersed submanifold of $\mathbb{R}^2$ when given an appropriate topology and smooth structure. As such, it is diffeomorphic to $\mathbb{R}^2$. But it is not an embedded submanifold of $\mathbb{R}^2$, because it does not have the subspace topology; see Exercise Sheet 9, Exercise 5(a).

Exercise: Let M be a smooth manifold and let $S \subseteq M$ be an immersed submanifold. Show that every subset of S that is open in the subspace topology is also open in its given submanifold topology, and the converse is true if and only if S is embedded.

Proposition 7.15. Let M be a smooth manifold and let S be an immersed submanifold of M. If any of the following holds, then S is embedded:

1. $\operatorname{codim}_M S = 0$,
2. The inclusion map $\iota : S \hookrightarrow M$ is proper,
3. S is compact.

Proof. Exercise Sheet 9, Exercise 1; follows readily from Proposition 6.6. □

Although many immersed submanifolds are not embedded, the next proposition shows that the local structure of an immersed structure is the same as that of an embedded one.

Proposition 7.16. (Immersed submanifolds are locally embedded) If M is a smooth manifold and $S \subseteq M$ is an immersed submanifold, then for each point $p \in S$ there exists a neighborhood U of p in S that is an embedded submanifold of M.

Proof. By assumption, $\iota : S \hookrightarrow M$ is a smooth immersion, so by the local embedding theorem Exercise Sheet 7, Exercise 3, every $p \in S$ has a neighborhood U in S such that $\iota|_U : U \hookrightarrow M$ is a smooth embedding, which proves the assertion. □

Finally, we discuss the tangent space to submanifolds. If S is a submanifold of $\mathbb{R}^n$, we intuitively think of the tangent space T_pS at a point $p \in S$ as a subspace of the tangent space $T_p\mathbb{R}^n$. Similarly, the tangent space to a smooth submanifold of an abstract smooth manifold can be viewed as a subspace of the tangent space to the ambient manifold, once we make appropriate identifications.

Let M be a smooth manifold and let S be an immersed or embedded submanifold of M. Since the inclusion map $\iota : S \hookrightarrow M$ is (at least) a smooth immersion, at each point $p \in S$ we have an injective linear map $d\iota_p : T_pS \to T_pM$. In terms of derivations, this injection works in the following way: for any vector $v \in T_pS$, the image vector $\tilde{v} = d\iota_p(v) \in T_pM$ acts on smooth functions on M by

$$\tilde{v}f = d\iota_p(v)(f) = v(f|_S).$$

We usually identify T_pS with its image $d\iota_p(T_pS)$ under $d\iota_p$, thereby thinking of T_pS as a certain linear subspace of T_pM. This identification makes sense regardless of whether S is embedded or immersed.

There are several alternative ways of characterizing T_pS as a subspace of T_pM; see Exercise Sheet 9, Exercise 3 and Exercise Sheet 9, Exercise 4 for such results. The next proposition, for instance, gives a useful way to characterize T_pS in the embedded case; one can show that it fails in the non-embedded case.

Proposition 7.17. Let M be a smooth manifold, let $S \subseteq M$ be an embedded submanifold and let $p \in S$. As a subspace of T_pM, the tangent space T_pS is characterized by

$$T_pS = \{v \in T_pM \mid vf = 0 \text{ whenever } f \in C^\infty(M) \text{ with } f|_S = 0\}.$$

Proof. Pick $v \in T_pS \subseteq T_pM$. Then $v = d\iota_p(w)$ for some $w \in T_pS$, where $\iota : S \hookrightarrow M$ is the inclusion map. If $f \in C^\infty(M)$ with $f|_S = 0$, then

$$vf = d\iota_p(w)(f) = w(f|_S) = 0.$$

Conversely, if $v \in T_pM$ satisfies $vf = 0$ whenever f vanishes on S, we have $v = d\iota_p(w)$ for some $w \in T_pS$. Let $(x^1, \dots, x^n)$ be slice coordinates for S in some neighborhood U of p, so that

$$U \cap S = \left\{(x^1, \dots, x^n) \in U \mid x^{k+1} = \dots = x^n = 0\right\},$$

and $(x^1, \dots, x^k)$ are coordinates for $U \cap S$. Since the inclusion map $\iota : U \cap S \hookrightarrow M$ has the coordinate representation

$$\iota(x^1, \dots, x^k) = (x^1, \dots, x^k, 0, \dots, 0),$$

in these coordinates, it follows that $T_pS \cong d\iota_p(T_pS)$ is exactly the subspace of T_pM spanned by

$$\left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^k}\right|_p.$$

If we write the coordinate representation of $v \in T_pM$ as

$$v = \sum_{i=1}^n v^i \left.\frac{\partial}{\partial x^i}\right|_p,$$

then $v \in T_pS$ if and only if $v^j = 0$ for all $j > k$.

Let φ be a smooth bump function supported in U that is equal to 1 in a neighborhood of p. Choose an index $j > k$ and consider the function $f(x) = \varphi(x)x^j$, extended to be zero on $M \setminus \operatorname{supp} \varphi$.

Then f vanishes identically on S, so

$$0 = vf = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i}\bigg|_p (\varphi(x)x^j) = v^j,$$

Thus, $v \in T_pS$, as desired. □

Given a smooth manifold M and a subset S of M, there are two very different questions one can ask. The simplest question is whether S is an embedded submanifold, since embedded submanifolds are exactly those subsets satisfying the local slice condition; this is simply a question about the subset S itself: either it is an embedded submanifold or it is not, and if so, then the topology and smooth structure making it into an embedded submanifold are uniquely determined according to Exercise Sheet 8, Exercise 6.

A more subtle question is whether S can be an immersed submanifold. In this case, neither the topology nor the smooth structure is known in advance, so one needs to ask whether there are any topology and smooth structure on S making it into an immersed submanifold. This question is not always straightforward to answer, and it can be especially tricky to prove that S is not an immersed submanifold. Here is an example of how this can be done.

Example 7.18. Consider the subset

$$S = \Big\{(x, y) \in \mathbb{R}^2 \mid y = |x|\Big\} \subseteq \mathbb{R}^2.$$

It is easy to check that $S \setminus \{(0, 0)\}$ is an embedded 1-dimensional submanifold of $\mathbb{R}^2$, so if S itself is an immersed submanifold at all, it must be 1-dimensional. Suppose there were some smooth manifold structure on S making it into an immersed submanifold. Then $T_{(0,0)}S$ would be a 1-dimensional subspace of $T_{(0,0)}\mathbb{R}^2$, so by Exercise Sheet 9, Exercise 3(a), there would be a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ whose image is in S, and that satisfies $\gamma(0) = (0, 0)$ and $\gamma'(0) \neq 0$. Writing $\gamma(t) = (x(t), y(t))$, we see that $y(t)$ takes a global minimum at $t = 0$, so $y'(0) = 0$. On the other hand, since every point $(x, y) \in S$ satisfies $x^2 = y^2$, we have $x(t)^2 = y(t)^2$ for all $t \in (-\varepsilon, \varepsilon)$. Differentiating twice and setting $t = 0$, we conclude that $2x'(0)^2 = 2y'(0)^2 = 0$, which is a contradiction. Thus, there is no such smooth manifold structure on S.

Addendum: Sard's Theorem and Whitney's Theorems

Theorem 7.19 (Sard's Theorem)**.** If $F : M \to N$ is a smooth map between smooth manifolds, then the set of critical values of F has measure zero in N.

- $\to$ "almost all" $c \in N$ are regular values of F
- $\Rightarrow$ "almost all" level sets $F^{-1}(c)$ of F are properly embedded submanifolds of M of dimension $\dim M - \dim N$.

Theorem 7.20 (Whitney's Embedding Theorem)**.** Every smooth n-manifold admits a proper smooth embedding into $\mathbb{R}^{2n+1}$.

$\rightarrow$ Every smooth n-manifold is diffeomorphic to a properly embedded submanifold of $\mathbb{R}^{2n+1}$.

Theorem 7.21 (Whitney's Immersion Theorem)**.** Every smooth n-manifold admits a smooth immersion into $\mathbb{R}^{2n}$.

The above two theorems are sometimes referred to as the easy or weak Whitney embedding and immersion theorems, because Whitney obtained later the following improvements.

Theorem 7.22 (Strong Whitney Embedding Theorem)**.** Given $n \geq 1$, every smooth n-manifold admits a smooth embedding into $\mathbb{R}^{2n}$.

Theorem 7.23 (Strong Whitney Immersion Theorem)**.** Given $n \geq 2$, every smooth n-manifold admits a smooth immersion into $\mathbb{R}^{2n-1}$.

For the proofs of all the above results, as well as a discussion of sets of measure zero (in $\mathbb{R}^n$ or in smooth manifolds), we refer to [Lee [1], Chapter 6 and Appendix C].

Chapter 8

Vector Bundles

In Chapter 5, we saw that the tangent bundle of a smooth manifold has a natural structure as a smooth manifold in its own right. The natural coordinates we constructed on TM make it look locally like the Cartesian product of an open subset of M^n with $\mathbb{R}^n$. This kind of structure arises quite frequently: a collection of vector spaces, one for each point in M, glued together in a way that looks locally like the Cartesian product of M with $\mathbb{R}^k$, but globally may be "twisted." Such structures are called vector bundles, and will be discussed briefly here.

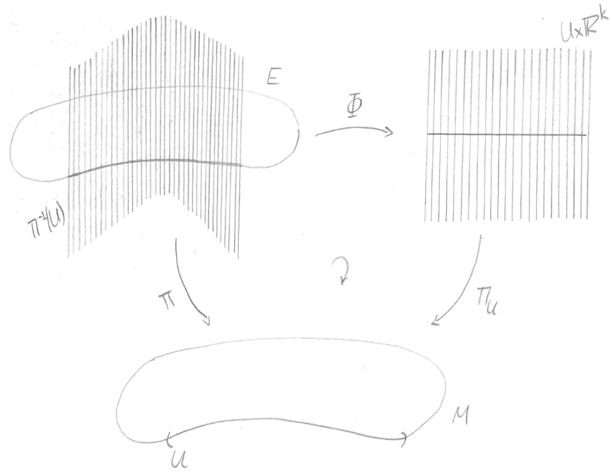
Definition 8.1. Let M be a topological space. A real vector bundle of rank k over M is a topological space E, together with a continuous surjective map $\pi : E \to M$, satisfying the following conditions:

(i) For each $p \in M$, the fiber $E_p = \pi^{-1}(p)$ over p is endowed with the structure of a k-dimensional $\mathbb{R}$-vector space.

(ii) For each $p \in M$, there exists an open neighborhood U of p in M and a homeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$, called a local trivialization of E over U, satisfying the following conditions:

- $\pi_U \circ \Phi = \pi$, where $\pi_U : U \times \mathbb{R}^k \to U$ is the projection.
- for each $q \in U$, the restriction of Φ to E_q is an $\mathbb{R}$-vector space isomorphism from E_q to $\{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

The space E is called the total space of the bundle, M is called its base, and π is its projection.



8.1 Smooth Vector Bundles

Definition 8.2. With the same notation as in Definition 8.1, if both M and E are smooth manifolds, π is a smooth map, and the local trivializations can be chosen to be diffeomorphisms, then E is called a smooth vector bundle over M. In this case, any local trivialization that is a diffeomorphism onto its image is called a smooth local trivialization.

Definition 8.3. With the same notation as in Definition 6.1, if there exists a local trivialization of E over all of M, called a global trivialization of E, then E is called a trivial bundle. If $E \to M$ is a smooth vector bundle that admits a smooth global trivialization, then we say that E is smoothly trivial. In this case, E is diffeomorphic to $M \times \mathbb{R}^k$, not just homeomorphic (as in the previous case).

Example 8.4. Given any topological space M, the product space $E = M \times \mathbb{R}^k$ with $\pi = \pi_M : M \times \mathbb{R}^k \to M$ as its projection is a rk k vector bundle over M. Any such bundle, called a product bundle, is trivial (with the identity map $\Phi = \mathrm{Id}_E : M \times \mathbb{R}^k \to M \times \mathbb{R}^k$ as a global trivialization). If M is a smooth manifold, then the (smooth) product bundle $M \times \mathbb{R}^k$ is smoothly trivial.

Proposition 8.5 (The tangent bundle as a vector bundle)**.** Let M be a smooth n-manifold and let TM be its tangent bundle. With its standard projection map $\pi : TM \to M$, its natural vector space structure on each fiber, and the topology and smooth structure constructed in Proposition 5.12, $\pi : TM \to M$ is a smooth vector bundle of rank n over M.

Proof. Given any smooth chart (U, φ) for M with coordinate functions (x^i), define a map

$$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k, \quad v^i \frac{\partial}{\partial x^i}\bigg|_p \mapsto (p, (v^1, \dots, v^n)).$$

This is linear on the fibers and satisfies $\pi_U \circ \Phi = \pi$. The composite map

$$\pi^{-1}(U) \xrightarrow{\Phi} U \times \mathbb{R}^k \xrightarrow{\varphi \times \mathrm{Id}_{\mathbb{R}^k}} \varphi(U) \times \mathbb{R}^k$$

is equal to the coordinate map $\tilde{\varphi} : \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^k$ constructed in Proposition 5.12, so it is a diffeomorphism (as a composition of diffeomorphisms). Thus, Φ satisfies all the conditions for a smooth local trivialization. $\square$

Any bundle that is not trivial requires more than one local trivialization. The next lemma shows that the composition of two smooth local trivializations has a simple form where they overlap.

Lemma 8.6. Let $\pi : E \to M$ be a smooth vector bundle of rk k over M. Suppose that

$$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k \quad \text{and} \quad \Psi : \pi^{-1}(V) \to V \times \mathbb{R}^k$$

are two smooth local trivializations of E with $U \cap V \neq \emptyset$. Then there exists a smooth map

$$\tau : U \cap V \to GL(k, \mathbb{R})$$

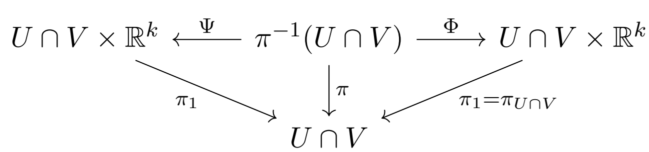
called <u>the transition function between the smooth local trivializations Φ and Ψ</u>, such that the composition

$$\Phi \circ \Psi^{-1} : U \cap V \ \times \ \mathbb{R}^k \to U \cap V \ \times \ \mathbb{R}^k$$

has the form

$$\Phi \circ \Psi^{-1}(p, v) = (p, \tau(p)v).$$

Proof. Note that the following diagram commutes:



and thus $\pi_1 \circ (\Phi \circ \Psi^{-1}) = \pi_1$, which means that

$$\Phi \circ \Psi^{-1}(p, v) = (p, \sigma(p, v))$$

for some smooth map $\sigma : U \cap V \ \times \ \mathbb{R}^k \to \mathbb{R}^k$. Moreover, for each fixed $p \in U \cap V$, the map $v \mapsto \sigma(p, v)$ is an invertible linear map (since both $\Phi|_{E_p}$ and $\Psi|_{E_p}$ are $\mathbb{R}$-linear isomorphisms), so there is an invertible $k \times k$ matrix $I(p)$ such that $\sigma(p, v) = \tau(p)v$. It remains to show that $\tau : U \cap V \to GL(k, \mathbb{R})$ is smooth; this is established in Exercise Sheet 10, Exercise 1(b). $\square$

Vector bundles are often most easily described by giving a collection of vector spaces, one for each point of the base manifold. In order to make such a set into a vector bundle, we would first have to construct a manifold topology and a smooth structure on the disjoint union of all the vector spaces, and then construct the local trivializations and show that they have the requisite properties. The next lemma provides a shortcut (c.f Lemma 2.9) by showing that it is sufficient to construct the local trivializations, as long as they overlap with smooth transition functions (see also Exercise Sheet 10, Exercise 2 for a stronger form of this result).

Lemma 8.7 (Vector bundle chart lemma)**.** Let M be a smooth manifold. Suppose that for each $p \in M$ we are given an $\mathbb{R}$-vector space E_p of some fixed dimension k. Set $E := \bigsqcup_{p \in M} E_p$, and consider the map $\pi : E \to M$, $v \in E_p \mapsto p$. Suppose furthermore that we are given the following data:

(i) an open cover $\{U_\alpha\}_{\alpha \in A}$ of M,

(ii) for each $\alpha \in A$, a bijective map $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ whose restriction to each E_p is an $\mathbb{R}$-vector space isomorphism from E_p to $\{p\} \times \mathbb{R}^k \cong \mathbb{R}^k$,

(iii) For each $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$, a smooth map $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k, \mathbb{R})$ such that the map $\Phi_\alpha \circ \Phi_\beta^{-1} : U_\alpha \cap U_\beta \ \times \ \mathbb{R}^k \to U_\alpha \cap U_\beta \ \times \ \mathbb{R}^k$ has the form

$$\Phi_\alpha \circ \Phi_\beta^{-1}(p, v) = (p, \tau_{\alpha\beta}(p)v).$$

Then E has a unique topology and smooth structure making it into a smooth manifold and a smooth vector bundle of rank k over M with π as projection and $\{(U_\alpha, \Phi_\alpha)\}_{\alpha \in A}$ as smooth local trivializations.

Proof. $\to$ [Lee [1], Lemma 10.6] □

Remark 8.8 (Restriction of a vector bundle)**.** Let $\pi : E \to M$ be a $\operatorname{rk} k$ vector bundle and let $S \subseteq M$ be any subset. We define the restriction of E to S to be the set $E|_S = \bigsqcup_{p \in S} E_p$, with the projection $E|_S \to S$ obtained by restricting π. If $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ is a local trivialization of E over $U \subseteq M$, it restricts to a bijective map from $(\pi|_S)^{-1}(U \cap S)$ to $(U \cap S) \times \mathbb{R}^k$, and it is easy to check that these form local trivializations for a vector bundle structure on $E|_S$.

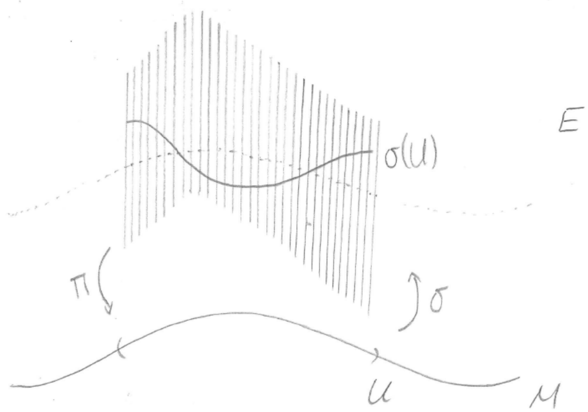
If E is a smooth vector bundle over M and $S \subseteq M$ is an embedded submanifold, it follows easily from Lemma 8.7 that $E|_S$ is a smooth vector bundle.

Finally, if E is a smooth vector bundle over M, but $S \subseteq$ is merely immersed, then we give $E|_S$ a topology and a smooth structure making it into a smooth $\operatorname{rk} k$ vector bundle over S as follows: For any $p \in S$, choose a neighborhood U of p in M over which there is a smooth local trivialization Φ of E, and a neighborhood V of p in S that is embedded in M and contained in U. Then the restriction of Φ to $\pi^{-1}(V)$ is a bijection from $\pi^{-1}(V)$ to $V \times \mathbb{R}^k$, and we can apply Lemma 8.7 to these bijections to yield the desired structure.

In particular, if $S \subseteq M$ is a smooth (immersed or embedded) submanifold, then $TM|_S$ is called the ambient tangent bundle over S.

8.2 Sections and Frames for Vector Bundles

Definition 8.9. Let $\pi : E \to M$ be a vector bundle. A local section of E is a continuous map $\sigma : U \to E$ defined on some open subset $U \subseteq M$ and satisfying $\pi \circ \sigma = \mathrm{Id}_U$.



This means that $\sigma(p) \in E_p$ for every $p \in U$. A global section of E is a section of E defined on all of M, i.e., a continuous map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{Id}_M$.

A rough (local or global) section of E over an open subset $U \subseteq M$ is defined to be a (not necessarily continuous) map $\sigma : U \to E$ such that $\pi \circ \sigma = \mathrm{Id}_U$. (Note that a local section of E over U is the same as a global section of the restricted bundle $E|_U$.)

The zero section of E is the global section $\zeta : M \to E$ defined by $\zeta(p) = 0 \in E_p$ for each $p \in M$.

If M is a smooth manifold and if E is a smooth vector bundle over M, then a smooth (local or global) section of is one that is a smooth map from its domain to E.

→ σ is cont/smooth: Exercise Sheet 10, Exercise 3(a).

→ sections of the product bundle: Exercise Sheet 10, Exercise 3(c).

If $E \to M$ is a smooth vector bundle, then the set of all smooth global sections of E is an $\mathbb{R}$-vector space under pointwise addition and scalar multiplication:

$$(c_1\sigma_1 + c_2\sigma_2)(p) = c_1\sigma_1(p) + c_2\sigma_2(p)$$

This vector space is usually denoted by $\Gamma(E)$ (but for particular vector bundles, we often introduce specialized notation for their spaces of global sections).

Smooth sections of $E \to M$ can be multiplied by smooth real-valued functions:

$$f \in C^\infty(M),\, \sigma \in \Gamma(E) \Rightarrow f\sigma \in \Gamma(E),\, (f\sigma)(p) = f(p)\sigma(p).$$

→ the various claims made above will be proved in Exercise Sheet 10, Exercise 3.

Lemma 8.10 (Extension lemma for vector bundles)**.** Let $\pi : E \to M$ be a smooth vector bundle. Let $A \subseteq M$ be a closed subset, and let $\sigma : A \to E$ be a section of $E|_A$ that is smooth in the sense that σ extends to a smooth local section of E in a neighborhood of each point. Then for each open subset $U \subseteq M$ containing A, there exists a smooth global section $\tilde{\sigma} \in \Gamma(E)$ such that $\tilde{\sigma}|_A = \sigma$ and $\mathrm{supp}(\tilde{\sigma})\left(= \overline{\{p \in M \mid \tilde{\sigma}(p) \neq 0\}}\right) \subseteq U$.

Proof. Exercise! (Similar to the proof of Lemma 4.5) ☐

→ see Exercise Sheet 10, Exercise 3(d) and Exercise Sheet 10, Exercise 4(d) for applications.

Definition 8.11. Let $E \to M$ be a vector bundle. If $U \subseteq M$ is an open subset, then a k-tuple of local sections $(\sigma_1, \dots, \sigma_k)$ of E over U is said to be linearly independent if their values $(\sigma_1(p), \dots, \sigma_k(p))$ form a linearly independent k-tuple in E_p for each $p \in U$. Similarly, they are said to span E if their values span E_p for each $p \in U$.

A local frame for E over U is an ordered k-tuple $(\sigma_1, \dots, \sigma_k)$ of linearly independent local sections of E over U that span E; thus, $(\sigma_1(p), \dots, \sigma_k(p))$ is a basis for E_p for each $p \in U$. It is called a global frame if $U = M$.

If, moreover, $E \to M$ is a smooth vector bundle, then a local or global frame for E is said to be smooth if each σ_i is a smooth section of E. (We often denote a frame $(\sigma_1, \dots, \sigma_k)$ by (σ_i).)

Example 8.12. (Global frame for a product bundle) If $E = M \times \mathbb{R}^k \to M$ is a (smooth) product bundle over a (smooth) manifold M, then the standard basis $(e_1, \dots, e_k)$ for $\mathbb{R}^k$ yields a (smooth) global frame $\tilde{e}_i$ for E, defined by

$$\tilde{e}_i : M \to E, \quad p \mapsto (p, e_i).$$

→ For the correspondence between smooth local frames and smooth local trivializations, see Exercise Sheet 10, Exercise 5 (which also settles the question of the existence of smooth local frames). See also Exercise Sheet 11, Exercise 1 (uniqueness of smooth structure on TM).

→ For the completion of smooth local frames of smooth vector bundles, see ES10E4.

We conclude this chapter with the important observation that smoothness of sections of vector bundles can be characterized in terms of local frames:

Assume that (σ_i) is a smooth local frame for E over some open subset $U \subseteq M$. If $\tau : M \to E$ is a rough section, the value of τ at an arbitrary point $p \in U$ can be written as

$$\tau(p) = \tau^i(p)\sigma_i(p)$$

for some uniquely determined numbers $(\tau^1(p), \dots, \tau^k(p))$. This defines k functions $\tau^i : U \to \mathbb{R}$, called the component functions of τ with respect to the given local frame (σ_i).

Proposition 8.13 (Local frame criterion for continuity/smoothness)**.** Let $\pi : E \to M$ be a continuous (respectively, smooth) vector bundle and let $\tau : M \to E$ be a rough section. If (σ_i) is a continuous (respectively, smooth) local frame for E over an open subset $U \subseteq M$, then τ is continuous (respectively, smooth) if and only if its component functions with respect to (σ_i) are continuous (respectively, smooth).

Proof. We prove the statement in the smooth case; the other case can be treated similarly. Let $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ be the smooth local trivialization associated with the smooth local frame; see Exercise Sheet 10, Exercise 5(a)(b). Since Φ is a diffeomorphism, τ is smooth on U if and only if $\Phi \circ \tau$ is smooth on U. By the construction in Exercise Sheet 10, Exercise 5(b), we know that

$$(\Phi \circ \tau)(p) = (p, (\tau^1(p), \dots, \tau^k(p))),$$

where (τ^i) are the component functions of τ with respect to (σ_i), so $\Phi \circ \tau$ is smooth if and only if the component functions τ^i are smooth according to Exercise Sheet 3, Exercise 4(b). $\square$

Note that the proposition applies equally well to local sections, since a local section of E over an open subset $V \subseteq M$ is a global section of the restricted bundle $E|_V$.

Chapter 9

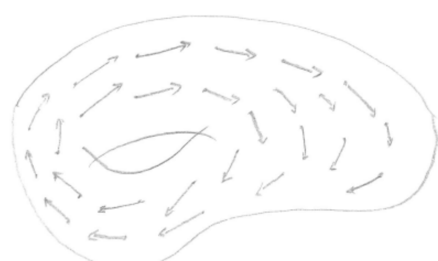
Vectors Fields and Flows

9.1 Vector Fields

Definition 9.1. A rough/continuous/smooth vector field on a smooth manifold M is a rough/continuous/ smooth (global) section of the tangent bundle TM.

If $U \subset M$ is open, the fact that π_U is naturally identified with T_pM for each point $p \in U$ (Proposition 5.9) allows us to identify TU with the open subset $\pi^{-1}(U) \subset TM$. Therefore, a vector field on U can be thought of either as a map $U \to TU$ or as a map $U \to TM$.

A vector field on an open subset $U \subset \mathbb{R}^n$ is simply a continuous map $U \to \mathbb{R}^n$, which can be visualized as attaching an "arrow" to each point of U. We visualize a vector field on an open subset U of a smooth manifold M in a similar way: as an arrow attached to each point of M, chosen to be tangent to M and to vary continuously from point to point.



The set $\mathfrak{X}(M)$ of all smooth (global) vector fields on a smooth manifold M is an infinite-dimensional $\mathbb{R}$-vector space and a module over the ring $C^\infty(M)$ (Exercise Sheet 10, Exercise 3).

→ $\exists$ extension lemma for vector fields: special case of lemma 8.10; see also Exercise Sheet 10, Exercise 3(d) for an application.

→ local/global frame for M = local/global frame for TM; see Definition 8.10.

→ completion of smooth local frames for M: special case of Exercise Sheet 10, Exercise 4.

Definition 9.2. Let M be a smooth manifold and let $X : M \to TM$ be a rough vector field on M. The support of X is defined as the closure of the set $\{p \in M \mid X_p \neq 0\}$. We say that X is compactly supported if its support is a compact set.

Let M and X be as above. If (U, x_i) is a smooth coordinate chart for M, then we can write the value of X at any point $p \in U$ in terms of the coordinate basis vectors:

$$X_p = X^i(p) \frac{\partial}{\partial x^i}\bigg|_p$$

This defines n functions $X^i : U \to \mathbb{R}$, called the component functions of X in the given chart.

Proposition 9.3 (Smoothness criterion for vector fields)**.** Let M be a smooth manifold and let $X : M \to TM$ be a rough vector field. If $(U, (x^i))$ is any smooth coordinate chart on M, then the restriction of X to U is smooth if and only if its component functions with respect to this chart are smooth.

Proof. Let $(U, (x^i))$ be a smooth chart for M and $(\pi^{-1}(U), (x^i, v^i) = \tilde{\varphi})$ be the natural coordinates on TM. The coordinate representation $\hat{X}$ of X with respect to these charts is

$$\begin{aligned} \hat{X}(x^1, \dots, x^n) &= \tilde{\varphi}\left(X^i(\varphi^{-1}(x)) \frac{\partial}{\partial x^i}\bigg|_{\varphi^{-1}(x)} \right) \\ &= (x^1, \dots, x^n, X^1(\varphi^{-1}(x)), \dots, X^n(\varphi^{-1}(x))) \end{aligned}$$

so X is smooth on U if and only if its component functions X^i, $1 \leq i \leq n$, are smooth on U. □

Example 9.4. 1) If (U, x_i) is any smooth chart on M, then the assignment $p \mapsto \frac{\partial}{\partial x^i}|_p$ determines a vector field on U, called the i-th coordinate vector field and denoted by $\frac{\partial}{\partial x^i}$. It is smooth by Proposition 9.3, because its component functions are constants.

In particular, the coordinate vector fields form a smooth local frame $\left(\frac{\partial}{\partial x^i}\right)$ for TM, called a coordinate frame. Note that every point of M is in the domain of such a local frame.

2) The Euler vector field on $\mathbb{R}^n$; see Exercise Sheet 11, Exercise 3.

An essential property of vector fields is that they define operators on the space of smooth real-valued functions. If $X \in \mathfrak{X}(M)$ and $f \in C^\infty(U)$, where $U \subset M$ is open, we obtain a new function $Xf : U \to \mathbb{R}$, defined by $p \mapsto (Xf)(p) := X_p f$. (Do not confuse the notations fX and Xf; the former is a smooth vector field on U obtained by multiplying X by f, while the latter is the real-valued function on U obtained by applying the vector field X to the smooth function f.) Since the action of a tangent vector on a function is determined by the values of the function in an arbitrarily small neighborhood (Proposition 5.8), it follows that Xf is locally determined. In particular, for any open subset $V \subset U$, we have $(Xf)|_V = X(f|_V)$.

This construction yields another useful smoothness criterion for vector fields (see also Proposition 8.13 for another one).

Proposition 9.5 (Smoothness criterion for vector fields)**.** : Let M be a smooth manifold and let $X : M \to TM$ be a rough vector field. Then the following are equivalent:

(a) X is smooth.

(b) $\forall f \in C^\infty(M)$, $Xf : M \to \mathbb{R}$ is smooth.

(c) $\forall U \subset M$ open, $\forall f \in C^\infty(U)$, $Xf : U \to \mathbb{R}$ is smooth.

Proof. (a) $\implies$ (b): Given $p \in M$, take a smooth chart $(U, (x^i))$ for M containing p. For $x \in U$ we may write

$$(Xf)(x) = \left(X^i(x) \frac{\partial}{\partial x^i}\bigg|_x \right) f = X^i(x) \frac{\partial f}{\partial x^i}(x).$$

Since the component functions X^i of X are smooth on U by Proposition 9.3, it follows that Xf is smooth on U. We conclude by Exercise Sheet 3, Exercise 9(a).

(b) $\implies$ (c): Fix $U \subseteq M$ open and $f \in C^\infty(U)$. For any point $p \in U$, let ψ be a smooth bump function that is equal to 1 in a neighborhood of p and supported in U (see Proposition 4.4), and define $\tilde{f} = \psi f$, extended to be zero on $M \setminus \operatorname{supp} \psi$. Then $X\tilde{f}$ is smooth by assumption, and is equal to Xf in a neighborhood of p (by the discussion on p. 92). We conclude by Exercise Sheet 3, Exercise 9(a).

(c) $\implies$ (a): If (x^i) are smooth local coordinates on $U \subseteq M$, then we can think of each coordinate x^i as a smooth function on U, and we have

$$X(x^i) = \left(X^j \frac{\partial}{\partial x^j}\bigg|_x \right)(x^i) = X^i,$$

which is smooth by assumption. We conclude by Proposition 9.3. □

One consequence of Proposition 9.5 is that a smooth vector field $X \in \mathfrak{X}(M)$ defines a map

$$C^\infty(M) \to C^\infty(M), \quad f \mapsto Xf,$$

which is $\mathbb{R}$-linear and satisfies the following product rule for vector fields:

$$X(fg) = fXg + gXf$$

(check this pointwise); in other words, this map is a derivation of $C^\infty(M)$.

The next proposition shows that derivations of $C^\infty(M)$ can be identified with smooth vector fields (and thus we sometimes use the same letter for both the smooth vector field (thought of as a map $M \to TM$) and the derivation of $C^\infty(M)$).

Proposition 9.6. Let M be a smooth manifold. A map $D : C^\infty(M) \to C^\infty(M)$ is a derivation if and only if it is of the form $Df = Xf$ for some $X \in \mathfrak{X}(M)$.

Proof. "$\Leftarrow$" We just showed above that any smooth vector field induces a derivation of $C^\infty(M)$.

"$\Rightarrow$" Let $p \in M$ and consider the map

$$X_p : C^\infty(M) \to \mathbb{R}, \quad f \mapsto (Df)(p).$$

Since D is $\mathbb{R}$-linear, X_p is also $\mathbb{R}$-linear, and since D is a derivation, we have

$$X_p(fg) = D(fg)(p) = (fD(g)+gD(f)(p) = f(p)D(g)(p)+g(p)D(f)(p) = f(p)X_pg+g(p)X_pf.$$

Hence, X_p is a derivation at $p \in M$, i.e., $X_p \in T_pM$. We obtain thus a rough vector field $X : M \to TM$, $p \mapsto X_p$, but since $Xf = Df$ is smooth for every $f \in C^\infty(M)$, X is actually smooth by Proposition 9.5, and we are done.

□

Utilizing Proposition 9.6, we now introduce an important way of combining two smooth vector fields to obtain another smooth vector field.

Let M be a smooth manifold and let $X, Y \in \mathfrak{X}(M)$. Given $f \in C^\infty(M)$, we can apply X to f to obtain $Xf \in C^\infty(M)$ (see Proposition 9.5) and we can now apply Y to Xf to obtain $Y(Xf) \in C^\infty(M)$. The operation $f \mapsto YXf$, though, does not satisfy the product rule in general, and thus cannot be a vector field (see Proposition 9.6), as the following example shows.

Example 9.7. Consider the vector fields

$$X = \frac{\partial}{\partial x} \quad \text{and} \quad Y = x\frac{\partial}{\partial y}$$

and the smooth functions

$$f(x, y) = x \quad \text{and} \quad g(x, y) = y.$$

On $\mathbb{R}^2$, we compute:

$$XY(fg) = X\left(x\frac{\partial(xy)}{\partial y}\right) = X(x^2) = \frac{\partial x^2}{\partial x} = 2x,$$

$$fXYg + gXYf = xX\underbrace{\left(x\frac{\partial y}{\partial y}\right)}_{=x} + yX\underbrace{\left(x\frac{\partial x}{\partial y}\right)}_{=0} = x(1) + y(0) = x$$

so XY is not a derivation of $C^\infty(\mathbb{R}^2)$.

We can also apply the same two vector fields in the opposite order, obtaining a (usually different) smooth function YXf. Applying both these operators to $f \in C^\infty(M)$ and subtracting, we obtain an operator:

$$[X, Y] : C^\infty(M) \to C^\infty(M), \quad f \mapsto XYf - YXf,$$

called the Lie bracket of X and Y. We will now show that $[X, Y]$ is a derivation of $C^\infty(M)$, and hence $[X, Y] \in \mathfrak{X}(M)$ by Proposition 9.6.

- $\mathbb{R}$-linearity: $[X, Y](af + bg) = a[X, Y]f + b[X, Y]g$ (follows from the $\mathbb{R}$-linearity of X and Y).

- Product rule:

$$\begin{aligned}
[X,Y](fg) &= XY(fg) - YX(fg) \\
&= X(fYg + gYf) - Y(fXg + gXf) \\
&= [(Xf)Yg + fXYg + (Xg)Yf + gXYf] - [(Yf)Xg + fYXg + (Yg)Xf + gYXf] \\
&= f(XYg - YXg) + g(XYf - YXf) \\
&= f[X,Y]g + g[X,Y]f.
\end{aligned}$$

→ for a geometric interpretation of the Lie bracket, see [Lee, §9.4 "Lie derivatives"].

→ for basic properties of the Lie bracket, see Exercise Sheet 11 and Exercise Sheet 12, Exercise 1.

If $S \subseteq M$ is an immersed or embedded submanifold, a vector field X on M does not necessarily restrict to a vector field on S, because $X_p \in T_pM$ may not lie in the subspace $T_pS \subseteq T_pM$ at a point $p \in S$. Given a point $p \in S$, a vector field X on M is said to be tangent to S at p if $X_p \in T_pS \subset T_pM$, and tangent to S if it is tangent to S at all points of S.

The following result is an immediate consequence of Proposition 7.17:

Proposition 9.8. Let M be a smooth manifold, $S \subseteq M$ be an embedded submanifold, and $X \in \mathfrak{X}(M)$. Then X is tangent to S if and only if $(Xf)|_S = 0$ for every $f \in C^\infty(M)$ such that $f|_S \equiv 0$.

9.2 Integral Curves and Flows

Let M be a smooth manifold. If $\gamma : J \subseteq \mathbb{R} \to M$ is a smooth curve, then for each $t \in J$, the velocity vector $\gamma'(t)$ is an element of $T_{\gamma(t)}M$. We describe next a way to work backwards: given a tangent vector at each point, we seek a curve whose velocity at each point is equal to the given vector there.

Definition 9.9. Let M be a smooth manifold and let V be a vector field on M. An integral curve of V is a differentiable curve $\gamma : J \to M$ whose velocity at each point is equal to the value of V at that point:

$$\gamma'(t) = V_{\gamma(t)}, \quad \forall t \in J.$$

If $0 \in J$, then $\gamma(0) \in M$ is called the starting point of γ.

Finding integral curves of vector fields boils down to solving a system of ODEs in a smooth chart: Suppose that $V \in \mathfrak{X}(M)$ and that $\gamma : J \to M$ is a smooth curve. On a smooth coordinate domain $U \subseteq M$, we can write γ in local coordinates as $\gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t))$. Then the condition $\gamma'(t) = V_{\gamma(t)}$ for γ to be an integral curve of V can be written:

$$\dot{\gamma}^i \frac{\partial}{\partial x^i}\bigg|_{\gamma(t)} = V^i(\gamma(t)) \frac{\partial}{\partial x^i}\bigg|_{\gamma(t)}$$

which reduces to the following autonomous system of ODEs:

$$\begin{cases} \dot{\gamma}^1(t) = V^1(\gamma^1(t), \dots, \gamma^n(t)) \\ \quad \vdots \\ \dot{\gamma}^n(t) = V^n(\gamma^1(t), \dots, \gamma^n(t)) \end{cases}$$

The fundamental fact about such systems is the following existence, uniqueness, and smoothness theorem:

Theorem Let $V : U \to \mathbb{R}^n$ be a smooth vector-valued function, where $U \subseteq \mathbb{R}^n$ is open. Consider the initial value problem

$$\dot{y}^i = V^i(y^1(t), \dots, y^n(t)), \quad 1 \leq i \leq n \tag{1}$$

$$y^i(t_0) = c^i, \quad 1 \leq i \leq n \tag{2}$$

for arbitrary $t_0 \in \mathbb{R}$ and $c = (c^1, \dots, c^n) \in \mathbb{R}^n$.

(a) Existence: For any $t_0 \in \mathbb{R}$ and $x_0 \in U$, there exists an open interval $J_0 \ni t_0$ and an open subset $x_0 \in U_0 \subseteq U$ such that for each $c \in U_0$, there is a C^1-map $y : J_0 \to U$ that solves (1)-(2).

(b) Uniqueness: Any two differentiable solutions to (1)-(2) defined on intervals containing t_0 agree on their common domain.

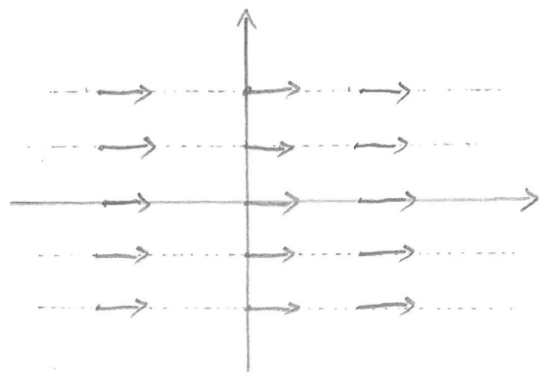
(c) Smoothness: Let J_0 and U_0 be as in (a), and consider the map $\Theta : J_0 \times U_0 \to U$, $(t, x) \mapsto y(t)$, where $y : J_0 \to U$ is the unique solution to (1) with initial condition $y(t_0) = x$. Then Θ is smooth.

An easy consequence of this theorem is the following result.

Proposition 9.10. Let V be a smooth vector field on a smooth manifold M. For each point $p \in M$, there exists $\varepsilon > 0$ and a (unique) smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ that is an integral curve of V starting at $p \in M$.

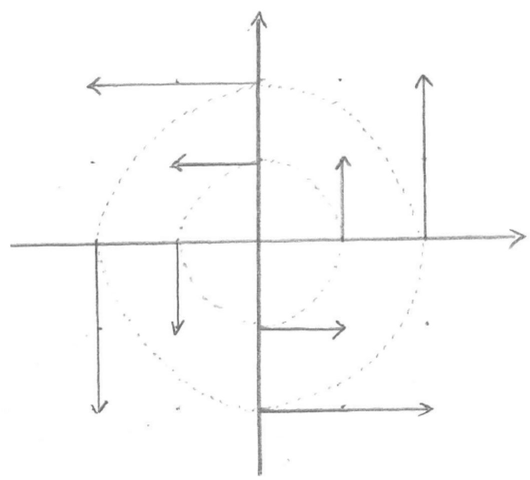
Example 9.11. Let (x, y) be the standard coordinates on $\mathbb{R}^2$.

1) Consider $V = \frac{\partial}{\partial x} \in \mathfrak{X}(\mathbb{R}^2)$. The integral curves of V are precisely the straight lines parallel to the x-axis, with parameterizations of the form $\gamma(t) = (a + t, b)$ for constants $a, b \in \mathbb{R}$.



Thus, there is a unique integral curve starting at each point of the plane, and the images of different integral curves are either identical or disjoint.

2) Consider $W = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2)$. To determine the integral curves of W, we proceed as follows (see p.97):



$$\gamma(t) = (\gamma^1(t), \gamma^2(t)) \quad \Rightarrow \quad \gamma'(t) = W_{\gamma(t)}$$

$$\Rightarrow \begin{cases} \dot\gamma_1(t) = -\gamma_2(t) \\ \dot\gamma_2(t) = \gamma_1(t) \end{cases} \xRightarrow{\ddot\gamma_1(t)+\gamma_1(t)=0}$$

$$\Rightarrow \begin{cases} \gamma_1(t) = a\cos t - b\sin t \\ \gamma_2(t) = a\sin t + b\cos t \quad (= -\dot\gamma_1(t)) \end{cases}$$

for constants $a, b \in \mathbb{R}$. Thus, each curve of the form $\gamma(t) = (a\cos t - b\sin t, a\sin t + b\cos t)$, $\quad t \in \mathbb{R}$ is an integral curve of W. When $(a, b) = (0, 0)$, this is the constant curve $\gamma(t) \equiv (0, 0)$; otherwise, it is a circle traversed clockwise. Since $\gamma(0) = (a, b)$, we see again that there is a unique integral curve starting at each point $(a, b) \in \mathbb{R}^2$, and the images of the various integral curves are either identical or disjoint.

Definition 9.12. Let M be a smooth manifold.

(a) A flow domain for M is an open subset $\mathcal{D} \subseteq \mathbb{R} \times M$ with the property that for each $p \in M$, the set

$$\mathcal{D}^{(p)} := \{t \in \mathbb{R} \mid (t, p) \in \mathcal{D}\} \subseteq \mathbb{R}$$

is an open interval containing $0 \in \mathbb{R}$.

(b) A flow on M is a continuous map $\Theta : \mathcal{D} \to M$, where $\mathcal{D} \subseteq \mathbb{R} \times M$ is a flow domain, which satisfies the following group laws:

- $\forall p \in M : \Theta(0, p) = p$

- $\forall s \in \mathcal{D}^{(p)}, \forall t \in \mathcal{D}^{(\Theta(s,p))} : s + t \in \mathcal{D}^{(p)}$ and we have $\Theta(t, \Theta(s, p)) = \Theta(s + t, p)$.

When $\mathcal{D} = \mathbb{R} \times M$ (and hence Θ is a continuous left $\mathbb{R}$-action on M), we say that Θ is a global flow on M (or a one-parameter group action).

(c) A maximal flow on M is a flow that admits no extension to a flow on a larger flow domain.

Let $\Theta : \mathcal{D} \to M$ be a flow on M.

- For each $p \in M$ we define the map

$$\Theta^{(p)} : \mathcal{D}^{(p)} \to M, \quad \Theta^{(p)}(t) = \Theta(t, p)$$

- For each $t \in \mathbb{R}$ we define the set

$$M_t := \{p \in M | (t, p) \in \mathcal{D}\}$$

and a map

$$\Theta_t : M_t \to M, \quad \Theta_t(p) := \Theta(t, p)$$

These maps satisfy $\Theta_t \circ \Theta_s = \Theta_{t+s}$ and $\Theta_0 = \mathrm{Id}_M$, so each Θ_t is in fact a diffeomorphism. Note that $p \in M_t \Leftrightarrow (t, p) \in \mathcal{D} \Leftrightarrow t \in \mathcal{D}^{(p)}$.

Proposition 9.13. If $\Theta : \mathcal{D} \to M$ is a smooth flow on M, then the infinitesimal generator V of Θ, defined as:

$$V : M \to TM, \quad p \mapsto V_p := \Theta^{(p)\prime}(0) = \left.\frac{d}{dt}\right|_{t=0} \Theta^{(p)}(t),$$

is a smooth vector field on M, and each curve $\Theta^{(p)}$ is an integral curve of V starting at p.

Proof. When $\mathcal{D} = \mathbb{R} \times M$, this is Exercise Sheet 12, Exercise 4. The proof of the general case is essentially identical to the proof for global flows. □

Theorem 9.14 (Fundamental theorem on flows)**.** Let V be a smooth vector field on a smooth manifold M. There is a unique smooth maximal flow $\Theta : \mathcal{D} \to M$ whose infinitesimal generator is V. This flow has the following properties:

(a) For each $p \in M$, the curve $\Theta^{(p)} : \mathcal{D}^{(p)} \to M$ is the unique maximal integral curve of V (maximal in the sense that it cannot be extended to an integral curve on any larger open interval) starting at p.

(b) If $s \in \mathcal{D}^{(p)}$, then $\mathcal{D}^{(\Theta(s,p))}$ is the interval

$$\mathcal{D}^{(\Theta(s,p))} = \mathcal{D}^{(p)} - s = \{t - s \mid t \in \mathcal{D}^{(p)}\}.$$

(c) For each $t \in \mathbb{R}$, the set M_t is open in M, and the map $\Theta_t : M_t \to M_{-t}$ is a diffeomorphism with inverse Θ_{-t}.

→ For the proof, we refer to [Lee [1], Theorem 9.12].

→ We have $p \in M_t \Rightarrow t \in \mathcal{D}^{(p)} \Rightarrow \mathcal{D}^{(\Theta(t,p))} = \mathcal{D}^{(p)} - t \Rightarrow -t \in \mathcal{D}^{(\Theta(t,p))} \Rightarrow \Theta_t(p) = \Theta(t,p) \in M_{-t}$, that is $\Theta_t : M_t \to M_{-t}$, $t \in \mathbb{R}$.

The flow whose existence and uniqueness are asserted in Theorem 9.14 is called the flow generated by V, or just the flow of V.

Example 9.15. The two vector fields on $\mathbb{R}^2$ described in Example 9.11 had integral curves defined for all $t \in \mathbb{R}$, so they generate global flows. We can write down their flows explicitly:

$$\Theta_V : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2, \quad (t,(x,y)) \mapsto (x+t, y)$$

(For each $t \in \mathbb{R} \setminus \{0\}, (\Theta_V)_t$ translates the plane to the left $(t < 0)$ or to the right $(t > 0)$ by a distance $|t|$).

$$\Theta_W : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2, \quad (t,(x,y)) \mapsto (x\cos t - y\sin t, x\sin t + y\cos t)$$

(For each $t \in \mathbb{R}$, $(\Theta_W)_t$ rotates the plane through an angle t about the origin.)

However, there are also smooth vector fields whose integral curves are not defined for all $t \in \mathbb{R}$. Here are two examples:

1) $M = \mathbb{R}^2 \setminus \{(0,0)\}, V := \frac{\partial}{\partial x} \in \mathfrak{X}(M)$. The unique integral curve of V starting at $(-1,0) \in M$ is $\gamma(t) = (t+1, 0)$. However, it cannot be extended continuously past $t = 1$ (this is intuitively evident because of the "hole" in M at the origin).

2) $M = \mathbb{R}^2, W := x^2 \frac{\partial}{\partial x} \in \mathfrak{X}(M)$. The unique integral curve of W starting at $(1,0)$ is $\gamma(t) = \left(\frac{1}{1-t}, 0\right)$. It cannot be extended past $t = 1$ because its x-coordinate is unbounded as $t \nearrow 1$.

Definition 9.16. A smooth vector field V on a smooth manifold M is called complete if it generates a global flow, or equivalently, if each of its maximal integral curves is defined for all $t \in \mathbb{R}$.

It is not always easy to determine by looking at a vector field whether it is complete or not. If one can solve the ODE explicitly to find all of the integral curves, and they all exist for all time, then the vector field is complete. On the other hand, if one can find one single integral curve that cannot be extended to all of $\mathbb{R}$, then it is not complete. However, it is often impossible to solve the ODE explicitly, so it is useful to have some general criteria for determining when a vector field is complete. The following theorem provides such a criterion. For its proof, we refer to [Lee [1], Theorem 9.16].

Theorem 9.17. Every compactly supported smooth vector field on a smooth manifold is complete.

In particular, on a compact smooth manifold, every smooth vector field is complete.

Chapter 10

Differential Forms

10.1 Covectors and Covector Fields

Definition 10.1. Let M be a smooth manifold. For each $p \in M$, we define the cotangent space at p, denoted by T_p^*M, to be the dual space to T_pM:

$$T_p^*M := (T_pM)^*.$$

Elements of T_p^*M are called (tangent) covectors at $p \in M$.

Given smooth local coordinates (x^i) on an open subset $U \subset M$, for each $p \in U$, the coordinate basis $\left(\frac{\partial}{\partial x^i}\big|_p\right)$ for T_pM gives rise to a dual basis for T_p^*M, which we denote temporarily by $(\lambda^i|_p)$. Any covector $\omega \in T_p^*M$ can thus be written uniquely as

$$\omega = w_i \lambda^i|_p,$$

where

$$\omega_i = \omega\left(\left.\frac{\partial}{\partial x^i}\right|_p\right).$$

Given now another set of smooth local coordinates $(\tilde{x}^j)$ whose domain contains $p \in U$, denote by $(\tilde{\lambda}^j|_p)$ the basis for T_p^*M dual to $\left(\frac{\partial}{\partial \tilde{x}^j}\big|_p\right)$. We can compute the components of the same covector $\omega \in T_p^*M$ with respect to the new coordinate system as follows. Recall first that the coordinate vector fields transform as follows:

$$\left.\frac{\partial}{\partial x^i}\right|_p = \frac{\partial \tilde{x}^j}{\partial x^i}(p)\left.\frac{\partial}{\partial \tilde{x}^j}\right|_p \qquad (*_1)$$

(see page 27). Writing ω in both systems as

$$\omega = \omega_i \lambda^i|_p = \tilde{\omega}_j \tilde{\lambda}^j|_p,$$

we can use $(*_1)$ to compute ω_i in terms of $\tilde{\omega}_j$:

$$\omega_i = \omega\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \omega\left(\frac{\partial \tilde{x}^j}{\partial x^i}(p)\left.\frac{\partial}{\partial \tilde{x}^j}\right|_p\right) = \frac{\partial \tilde{x}^j}{\partial x^i}(p)\,\omega\left(\left.\frac{\partial}{\partial \tilde{x}^j}\right|_p\right) = \frac{\partial \tilde{x}^j}{\partial x^i}(p)\tilde{\omega}_j \qquad (*_2)$$

Definition 10.2. Let M be a smooth manifold. The cotangent bundle of M is denoted by T^*M and is defined as the disjoint union:

$$T^*M := \bigsqcup_{p\in M} T_p^*M.$$

There is a natural projection map

$$\pi : T^*M \to M, \quad \omega \in T_p^*M \mapsto p.$$

As above, given any smooth local coordinates (x^i) on an open subset $U \subset M$, for each $p \in U$ we denote by $(\lambda^i|_p)$ the basis for T_p^*M dual to $\left(\frac{\partial}{\partial x^i}|_p\right)$. This defines n maps

$$\lambda^1, \dots, \lambda^n : U \to T^*M$$

(to be denoted differently soon) called coordinate covector fields.

Proposition 10.3 (The cotangent bundle as a vector bundle)**.** Let M be a smooth n-manifold. With its standard projection map and the natural vector space structure on each fiber, the cotangent bundle T^*M has a unique topology and smooth structure making it into a smooth vector bundle of rank n over M, for which all coordinate covector fields are smooth local sections.

Proof. (Similar to the proof of Proposition 8.5) Given a smooth chart (U, φ) for M, with coordinate s (x^i), define

$$\begin{aligned} \Phi : \pi^{-1}(U) &\to U \times \mathbb{R}^n \\ \xi_i \lambda^i|_p &\mapsto (p, (\xi_1, \dots, \xi_n)) \end{aligned}$$

where λ^i is the i-th coordinate covector field associated with (x^i). Suppose that $(\tilde{U}, \tilde{\varphi})$ is another smooth chart for M with coordinate s $(\tilde{x}^j)$, and let $\tilde{\Phi} : \pi^{-1}(\tilde{U}) \to \tilde{U} \times \mathbb{R}^n$ be defined analogously. On $\pi^{-1}(U \cap \tilde{U})$, it follows from $(*_2)$ that

$$\left(\Phi \circ \tilde{\Phi}^{-1}\right)\left(p, (\tilde{\xi}_1, \dots, \tilde{\xi}_n)\right) = \left(p, \left(\frac{\partial \tilde{x}^j}{\partial x^1}(p)\tilde{\xi}_j, \dots, \frac{\partial \tilde{x}^j}{\partial x^n}(p)\tilde{\xi}_j\right)\right)$$

The $GL(n, \mathbb{R})$-valued $(\partial \tilde{x}^j / \partial x^i)$ is smooth, so it follows from the vector bundle chart lemma (Lemma 8.7) that T^*M has a smooth structure making it into a smooth vector bundle for which the maps Φ are local trivializations. Uniqueness follows as in the proof of Exercise Sheet 11, Exercise 1. □

As in the case of the tangent bundle, smooth local coordinates for M yield smooth local coordinates for its cotangent bundle. If (x^i) are smooth coordinates on an open subset $U \subset M$, Exercise Sheet 10, Exercise 5(d) shows that the map

$$\pi^{-1}(U) \to \mathbb{R}^{2n}, \quad \xi_i \lambda^i|_p \mapsto (x^1(p), \dots, x^n(p), \xi_1, \dots, \xi_n)$$

is a smooth coordinate chart for T^*M. We call (x^i, ξ_i) the natural coordinates for T^*M associated with (x^i).

Definition 10.4. A rough/cont./smooth (local or global) section of T^*M is called a rough/cont./smooth covector fi or a (differential) 1-form on a smooth manifold M.

The set $\mathfrak{X}^*(M)$ of all smooth (global) covector fields on a smooth manifold M is an infinite-dimensional $\mathbb{R}$-vector space and a module over the ring $C^\infty(M)$ (Exercise Sheet 10, Exercise 3).

→ Local/global coframe for M = local/global frame for T^*M; see Definition 8.11.

→ Completion of smooth local coframes for M: special case of Exercise Sheet 10, Exercise 4.

Example 10.5. For any smooth chart $(U, (x^i))$, the coordinate covector fields (λ^i) defined above constitute a local coframe over U, called a coordinate coframe. By Exercise Sheet 13.I, Exercise 1 = Proposition 10.6, every coordinate coframe is smooth, because its component s in the given chart are constants.

In any smooth local coordinates (x^i) on an open subset $U \subseteq M$, a (rough) covector field ω can be written in terms of the coordinate covector fields (λ^i) as $\omega = \omega_i \lambda^i$ for n-functions $\omega_i : U \to \mathbb{R}$, called the component s of ω in the given chart and characterized by:

$$\omega_i(p) = \omega_p \left(\frac{\partial}{\partial x^i}\bigg|_p \right)$$

If ω is a (rough) covector field and if X is a (rough) vector field on M, then we can form a function

$$\omega(X) : M \to \mathbb{R}, \quad p \mapsto \omega_p(X_p)$$

If we write $\omega = \omega_i \lambda^i$ and $X = X^i \partial_i$ in terms of local coordinates, then $\omega(X)$ has the local coordinate representation

$$\omega(X) = \omega_i X^i$$

Just as in the case of vector fields (see Proposition 9.3 and 9.5), there are several ways to check smoothness of covector fields.

Proposition 10.6 (Smoothness criterion for covector fields)**.** Let M be a smooth manifold and let $\omega : M \to T^*M$ be a rough covector field. Then the following are equivalent:

(a) ω is smooth.

(b) In every smooth chart, the component functions of ω are smooth.

(c) Each point of M is contained in some coordinate chart in which ω has smooth component functions.

(d) For every $X \in \mathfrak{X}(M)$, the function $\omega(X) : M \to \mathbb{R}$ is smooth.

(e) For every open subset $U \subseteq M$ and every smooth vector field X on U, the function $\omega(X) : U \to \mathbb{R}$ is smooth.

Proof. Exercise Sheet 13.I, Exercise 1. □

Since any open subset of a smooth manifold is again a smooth manifold, Proposition 10.6 applies equally well to covector fields defined only on some open subset of M.

→ ∃ local coframe criterion for continuity/smoothness of rough covector fields: special case of Proposition 8.13.

The most important application of covector fields is that they enable us to interpret in a coordinate-independent way the partial derivatives of a smooth as the components of a covector field.

Let $f \in C^\infty(M)$. We define a covector field df, called the differential of f at $p \in M$, by

$$df_p(v) = vf, \quad v \in T_pM$$

Proposition 10.7. The differential of a smooth is a smooth covector field.

Proof. It is straightforward to check that $df_p \in T_p^*M$ for all $p \in M$. To verify that df is smooth, we apply Proposition 10.6(d): for any $X \in \mathfrak{X}(M)$, the $df(X)$ is smooth, because it is equal to Xf (see Proposition 9.5). □

For a smooth real-valued function $f : M \to \mathbb{R}$, we now have two different definitions for the differential of f at $p \in M$. In Ch. 3, we defined df_p as a linear map $T_pM \to T_{f(p)}\mathbb{R}$, while here we defined df_p as a covector at $p \in M$, i.e. a linear map $T_pM \to \mathbb{R}$. These are really the same object, once we take into account the identification between $T_{f(p)}\mathbb{R}$ and $\mathbb{R}$; one easy way to see this is to note that both are represented in coordinates by the row matrix whose components are the partial derivatives of f. (Let us verify this below for df defined as above.)

Let us compute the coordinate representation of df. Let (x^i) be smooth coordinates on an open subset $U \subseteq M$ and let (λ^i) be the corresponding coordinate coframe on U. Write df in coordinates as $df_p = A_i(p)\lambda^i|_p$ for some functions $A_i : U \to \mathbb{R}$. Then the definition of df implies

$$A_i(p) = df_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \frac{\partial}{\partial x^i}\Big|_p f = \frac{\partial f}{\partial x^i}(p),$$

which yields the following formula for the coordinate representation of df:

$$df_p = \frac{\partial f}{\partial x^i}(p)\lambda^i\Big|_p. \qquad (*_3)$$

Thus, the component functions of df in any smooth coordinate chart are the partial derivatives of f with respect to those coordinates. Due to this, we can think of df as an analogue of the classical gradient (the vector field in $\mathbb{R}^n$ whose components are the partial derivatives of the function), reinterpreted in a way that makes coordinate-independent sense on a manifold.

If we apply $(*_3)$ to the special case in which f is one of the coordinate functions $x^i : U \to \mathbb{R}$, we obtain

$$dx^j\Big|_p = \frac{\partial x^j}{\partial x^i}(p)\lambda^i\Big|_p = \delta^j_i \lambda^i\Big|_p = \lambda^j\Big|_p,$$

in other words, the coordinate vector field λ^j is none other than the differential dx^j. Therefore, $(*_3)$ can be rewritten as

$$df_p = \frac{\partial f}{\partial x^i}(p) dx^i\Big|_p, \quad p \in U,$$

or as an equation between covector fields instead of covectors:

$$df = \frac{\partial f}{\partial x^i} dx^i. \tag{$*_4$}$$

In particular, in the 1-dimensional case, this reduces to

$$df = \frac{df}{dx} dx.$$

Thus, we have recovered the familiar classical expression for the differential of a function f in coordinates. Henceforth, we abandon the notation λ^i for the coordinate coframe, and use dx^i instead.

Example 10.8. If $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto x^2 y \cos x$, then

$$df = \frac{\partial(x^2 y \cos x)}{\partial x} dx + \frac{\partial(x^2 y \cos x)}{\partial y} dy = (2xy \cos x - x^2 y \sin x)dx + (x^2 \cos x)dy.$$

→ properties of differential: Exercise Sheet 13.I, Exercise 2.

→ Derivative of a function along a curve: Exercise Sheet 13.I, Exercise 3(a)

10.2 Pullback of Covectors

Definition 10.9. Let $F : M \to N$ be a smooth map and let $p \in M$. The differential $dF_p : T_pM \to T_{F(p)}N$ yields a dual linear map, $dF_p^* : T^*_{F(p)}N \to T^*_pM$, called the <u>(pointwise) pullback by F at p</u> (or the cotangent map of F) and characterized by

$$dF_p^*(\omega)(v) = \omega(dF_p(v)), \quad \omega \in T^*_{F(p)}N, \quad v \in T_pM.$$

Unlike vector fields, whose pushforwards are only defined in special cases (see for example Exercise Sheet 11, Exercise 4), covector fields always pullback to covector fields.

Definition 10.10. Let $F : M \to N$ be a smooth map and let $\omega : N \to T^*N$ be a rough covector field. We define a rough covector field $F^*\omega$ on M, called the pullback of ω by F, by

$$(F^*\omega)_p := dF_p^*(\omega_{F(p)}) \tag{$*_5$}$$

It acts on a vector $v \in T_pM$ by

$$(F^*\omega)_p(v) = \omega_{F(p)}(dF_p(v)).$$

Proposition 10.11. Let $F : M \to N$ be a smooth map and let ω be a covector field on N. If $u : N \to \mathbb{R}$ is a continuous function, then

$$F^*(u\omega) = (u \circ F)F^*\omega.$$

If additionally u is smooth, then

$$F^*(du) = d(u \circ F).$$

Proof. We have

$$\begin{aligned} F^*(u\omega)_p = dF_p^*\big((u\omega)_{F(p)}\big) &= dF_p^*\big(u(F(p))\omega_{F(p)}\big) = u(F(p))dF_p^*(\omega_{F(p)}) \\ &= (u \circ F)(p)(F^*\omega)_p = \Big((u \circ F)(F^*\omega)\Big)(p) \end{aligned}$$

which proves the first statement. Now, for the second statement, if $p \in M$ and $v \in T_pM$, then

$$(F^*(du))_p(v) = dF_p^*\big((du)_{F(p)}\big)(v) = du_{F(p)}(dF_p(v)) = dF_p(v)u = v(u \circ F) = d(u \circ F)_p(v),$$

which yields the second statement. □

Proposition 10.12. Let $F : M \to N$ be a smooth map and let ω be a (continuous) covector field on N. Then $F^*\omega$ is a (continuous) covector field on M, and if ω is smooth, then so is $F^*\omega$.

Proof. Fix $p \in M$ and choose smooth coordinates (y^j) for N in a neighborhood V of $F(p)$. Set $U := F^{-1}(V)$ and observe that U is a neighborhood of p in M. Writing ω in coordinates as $\omega = w_j dy^j$ for (continuous) functions on V and using Proposition 10.11 twice (for $F|_U$), we compute that

$$F^*\omega = F^*(w_j dy^j) = (w_j \circ F)F^* dy^j = (w_j \circ F)d(y^j \circ F). \tag{$*_6$}$$

This expression is continuous, and it is smooth when ω is smooth, so we are done. □

Formula ($*_6$) for the pullback of a covector field can also be written in the following way

$$F^*\omega = (\omega_j \circ F)d(y^j \circ F) = (\omega_j \circ F)dF^j$$

where F^j is the j-th component function of F in these coordinates. Using either of these formulas, the computation of pullbacks in coordinate is quite simple.

Example 10.13. Consider the smooth map

$$F : \mathbb{R}^3 \to \mathbb{R}^2, \quad (x, y, z) \mapsto (x^2 y, y \sin z) = (u, v)$$

and the smooth covector field

$$\omega = udv + vdu \in \mathfrak{X}^*(\mathbb{R}^2).$$

According to $(*_6)$, we have

$$\begin{aligned} F^*\omega &= (u \circ F)d(v \circ F) + (v \circ F)d(u \circ F) \\ &= (x^2 y)d(y \sin z) + (y \sin z)d(x^2 y) \\ &= (x^2 y)(\sin z dy + y \cos z dz) + (y \sin z)(2xy dx + x^2 dy) \\ &= (2xy^2 \sin z)dx + (2x^2 y \sin z)dy + (x^2 y^2 \cos z)dz \end{aligned}$$

In other words, to compute $F^*\omega$, all we need to do is substitute the component facts of F for the coordinate facts of N everywhere they appear in ω.

→ see also [Lee [1], Example 11.28] for an example about the transformation law for a covector field under a change of coordinates.

In Ch.7 (see also Exercise Sheet 12, Exercise 1) we considered the conditions under which a vector field restricts to a submanifold The restriction of covector fields to submanifolds is much simpler and will be briefly discussed below (see also ES13-I-E3(a)).

Let M be a smooth manifold, let $S \subseteq M$ be an immersed submanifold and let $\iota : S \hookrightarrow M$ be the inclusion map. If $\omega \in \mathfrak{X}^*(M)$, then $\iota^*\omega \in \mathfrak{X}^*(S)$. More precisely, given $p \in S$ and $v \in T_pS$, we have

$$(\iota^*\omega)_p v = \omega_p(d\iota_p(v)) = \omega_p(v),$$

since $d\iota_p : T_pS \hookrightarrow T_pM$ is just the inclusion map under our usual identification of T_pS with the subspace $d\iota_p(T_pS)$ of T_pM. Thus, $\iota^*\omega$ is just the restriction of ω to vectors tangent to S. For this reason, $\iota^*\omega$ is often called <u>the restriction of ω to S</u>. Note, however, that $\iota^*\omega$ might equal zero at a given pt of S, even though considered as a covector field on M, ω might not vanish there. For example:

Example 10.14. Consider $\omega = dy \in \mathfrak{X}^*(\mathbb{R}^2)$ and let $S = \{y = 0\}$ be the x-axis, considered as an embedded submanifold of $\mathbb{R}^2$. As a covector field on $\mathbb{R}^2$, ω is clearly nonzero everywhere, because one of its components is always equal to 1. However, the restriction $\iota^*\omega$ of ω to S is identically zero, because dy vanishes identically on S:

$$\iota^*\omega = \iota^* dy = d(y|_S) = 0.$$

To distinguish the two ways in which we might interpret the statement "ω vanishes on S", one usually says that ω vanishes along S (or vanishes at points of S) if $\omega_p = 0$ for every $p \in S$. The weaker condition that $\iota^*\omega = 0$ is expressed by saying that the restriction of ω to S vanishes (or the pullback of ω to S vanishes).

Given a vector bundle $\pi_E : E \to M$, a subbundle of E is a vector bundle $\pi_D : D \to M$, in which D is a topological subspace of E and π_D is the restriction of π_E to D, such that for each $p \in M$, the subset $D_p = D \cap E_p$ is a linear subspace of E_p, and the vector space structure on D_p is the one inherited from E_p. Note that the condition that D be a vector bundle over M implies that all of the fibers D_p are non-empty and have the same dimension.

If $E \to M$ is a smooth vector bundle, then a subbundle of E is called a smooth subbundle if it is a smooth vector bundle and an embedded submanifold of E.

The following lemma [Lee [1], Lemma 10.32] gives a convenient condition for checking that a union of subspaces $\{D_p \subseteq E_p | \, p \in M\}$ is a smooth subbundle.

Lemma (Local frame criterion for subbundles)**.** Let $\pi : E \to M$ be a smooth vector bundle. Suppose that for each $p \in M$ we are given an m-dimensional linear subspace $D_p \subseteq E_p$. Then $D = \cup_{p\in M} D_p \subseteq E$ is a smooth subbundle of E if and only if the following condition is satisfied:

> "Each pt of M has a neighborhood U on which there exist smooth local sections $\sigma_1, \ldots, \sigma_m : U \to E$ with the property that $\sigma_1(q), \ldots, \sigma_m(q)$ form a basis for D_q at each $q \in U$."

10.3 Differential Forms

Definition 10.15. Let M be a smooth manifold.

(a) We define the bundle of covariant k-tensors on M by

$$T^k(T^*M) := \bigsqcup_{p\in M} T^k(T_p^*M)$$

with the obvious projection map, which is often also called a tensor bundle over M. Its sections are called covariant k-tensor fields on M.

(b) The subset of $T^k(T^*M)$ consisting of alternating k-tensors is denoted by $\Lambda^k(T^*M)$:

$$\Lambda^k(T^*M) := \bigsqcup_{p\in M} \Lambda^k(T_p^*M).$$

It can be shown (exercise!) that $\Lambda^k(T^*M)$ is a smooth subbundle of $T^k(T^*M)$, and thus it is a smooth vector bundle of rank $\binom{n}{k}$ over M. Its sections are called differential k-forms on M; they are (continuous) tensor fields whose value at each pt is an alternating k-tensor. The integer k is called the degree of the form. We denote the vector space of smooth k-forms by

$$\Omega^k(M) = \Gamma(\Lambda^k(T^*M)).$$

Note that a 0-form is just a continuous real-valued function on M (because $\Lambda^0(T^*M) = \bigsqcup_{p\in M} \Lambda^0(T_p^*M) = \bigsqcup_{p\in M} \mathbb{R} = M \times \mathbb{R}$, see Exercise Sheet 10, Exercise 3(c)), and a 1-form is a covector field on M (since $\Lambda^1(T^*M) = \bigsqcup_{p\in M} \Lambda^1(T_p^*M) \cong \bigsqcup_{p\in M} T_p^*M = T^*M$).

The wedge product of two differential forms is defined pointwise:

$$(\omega \wedge \eta)_p := \omega_p \wedge \eta_p$$

Thus, the wedge product of a k-form with an l-form is a $(k+l)$-form. If f is a 0-form and η is a k-form, then we interpret the wedge product $f \wedge \eta$ to mean the ordinary product $f\eta$. If we define

$$\Omega^*(M) := \bigoplus_{k=0}^{n} \Omega^k(M),$$

then the wedge product turns $\Omega^*(M)$ into an associative, anti-commutative, graded $\mathbb{R}$-algebra.

In any smooth chart $(U, (x^i))$, a k-form ω can be written as

$$\omega = \sum_I{}' \omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \sum_I{}' \omega_I dx^I,$$

where the coefficients ω_I are smooth functions defined on the coordinate domain U, and we use dx^I as an abbreviation for $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ (where $I = (i_1, \ldots, i_k)$), and the primed summation sign denotes a sum over only increasing multi-indices. According to Proposition 8.13, ω is smooth if and only if the component functions ω_I are smooth. Since

$$dx^{i_1} \wedge \cdots \wedge dx^{i_k} \left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}} \right) = \delta^I_J,$$

see [Multilinear Algebra, Lemma 19(c)], the component functions ω_I of ω are determined by

$$\omega_I = \omega \left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}} \right)$$

If $F : M \to N$ is a smooth map and ω is a differential form on N, the $F^*\omega$ is a differential form on

M, defined as follows

$$(F^*\omega)_p(v_1, \ldots, v_k) := \omega_{F(p)}\left(dF_p(v_1), \ldots, dF_p(v_k)\right)$$

Lemma 10.16. Let $F : M \to N$ be a smooth map. The following statements hold:

(a) $F^* : \Omega^k(N) \to \Omega^k(M)$ is linear over $\mathbb{R}$.

(b) $F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$

(c) In any smooth coordinates chart $(V, (y^i))$ for N, we have

$$F^*\left(\sum_I \omega_I \, dy^{i_1} \wedge \cdots \wedge dy^{i_k}\right) = \sum_I (\omega_I \circ F)\, d(y^{i_1} \circ F) \wedge \cdots \wedge d(y^{i_k} \circ F)$$

Proof. Exercise Sheet 13.II, Exercise 1. □

This lemma gives a computational rule for pullbacks of differential forms similar to the one we developed earlier for covector fields.

Example 10.17. Consider the smooth function

$$F : \mathbb{R}^2 \to \mathbb{R}^3, \quad (u, v) = (u, v, u^2 - v^2)$$

and the smooth 2-form

$$\omega = ydx \wedge dz + xdy \wedge dz \in \Omega^2(\mathbb{R}^3)$$

Then

$$\begin{aligned}
F^*\omega &= F^*(ydx \wedge dz + xdy \wedge dz) \\
&= vdu \wedge d(u^2 - v^2) + udv \wedge d(u^2 - v^2) \\
&= vdu \wedge (2udu - 2vdv) + udv \wedge (2udu - 2vdv) \\
&= -2v^2 du \wedge dv + 2u^2 dv \wedge du \\
&= -2(u^2 + v^2) du \wedge dv
\end{aligned}$$

→ see also Exercise Sheet 13.II, Exercise 2 for an example regarding the change of coordinates.

Proposition 10.18 (Pullback formula for top degree forms)**.** Let $F : M \to N$ be a smooth map between smooth n-manifolds. If (x^i) and (y^j) are smooth coordinates on open subsets $U \subseteq M$ and $V \subseteq N$, respectively, and u is a continuous real-valued function on V, then the following holds on $U \cap F^{-1}(V)$:

$$F^*(u\, dy^1 \wedge \cdots \wedge dy^n) = (u \circ F) \det DF \,(dx^1 \wedge \cdots \wedge dx^n) \qquad (*_7)$$

where DF represents the Jacobian matrix of F in these coordinates.

Proof. Since the fiber of $\Lambda^n(T^*M)$ is spanned by $dx^1 \wedge \cdots \wedge dx^n$ at each point, it suffices to show that both sides of $(*_7)$ agree when evaluated on $(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})$. We have

$$F^*(udy^1 \wedge \cdots \wedge dy^n) = (u \circ F)d(y^1 \circ F) \wedge \cdots \wedge d(y^n \circ F)$$

$$\begin{aligned}
\implies F^*(udy^1 \wedge \cdots \wedge dy^n)\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right) &= (u \circ F)dF^1 \wedge \cdots \wedge dF^n \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right) \\
&= (u \circ F)\det\left(dF^j\left(\frac{\partial}{\partial x^i}\right)\right) \\
&= (u \circ F)\det\left(\frac{\partial F^j}{\partial x^i}\right) \cdot 1 \\
&= F^*(u)\det DF\,(dx^1 \wedge \cdots \wedge dx^n)\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right)
\end{aligned}$$

with $1 = (dx^1 \wedge \cdots \wedge dx^n)\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right)$

as desired. □

Corollary 10.19. If $(U, (x^i))$ and $(\tilde{U}, (\tilde{x}^i))$ are overlapping smooth coordinate charts on M, then the following identity holds on $U \cap \tilde{U}$:

$$d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^n = \det\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) dx^1 \wedge \cdots \wedge dx^n.$$

Proof. Apply Proposition 10.18 for $F = \mathrm{Id}_{U \cap \tilde{U}}$, but using coordinates (x^i) in the domain and $(\tilde{x}^i)$ in the codomain. □

10.4 The Exterior Derivative

We now define a natural differential operator on smooth forms, called the exterior derivative, which is a generalization of the differential of a function. More precisely, for each smooth manifold M, we will show that there is a differential operator $d : \Omega^k(M) \to \Omega^{k+1}(M)$ satisfying $d \circ d = 0$ for all ω.

The definition of d on Euclidean space is straightforward: if $\omega = \sum_J \omega_J dx^J$ is a smooth k-form on an open subset $U \subset \mathbb{R}^n$, we define its exterior derivative $d\omega$ to be the following $(k+1)$-form:

$$d\left(\sum_J \omega_J dx^J\right) = \sum_J d\omega_J \wedge dx^J \qquad (*_8)$$

where $d\omega_J$ is the differential of the smooth function ω_J. In somewhat more detail, this is

$$d\left(\sum_J \omega_J dx^{j_1} \wedge \cdots \wedge dx^{j_k}\right) = \sum_J \sum_i \frac{\partial \omega_J}{\partial x^i} dx^i \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}.$$

For instance, for a smooth 0-form f we have

$$df = \sum_i \frac{\partial f}{\partial x^i} dx^i,$$

which is just the differential of f (see $*_4$), and for a smooth 1-form ω we compute that

$$d\omega = \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} \right) dx^i \wedge dx^j.$$

In order to transfer this definition to manifolds, we first need to check that it satisfies the following properties.

Proposition 10.20 (Properties of the exterior derivative on $\mathbb{R}^n$)**.** The exterior derivative has the following properties:

(a) d is $\mathbb{R}$-linear.

(b) If ω is a smooth k-form and η is a smooth l-form on an open subset $U \subseteq \mathbb{R}^n$, then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(c) $d \circ d \equiv 0$.

(d) d commutes with pullbacks: if $F : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^m$ is a smooth map, and $\omega \in \Omega^k(V)$, then

$$F^*(d\omega) = d(F^*\omega).$$

Proof. (a) Follows immediately from the definition.

(b) Due to (a), it suffices to consider terms of the form $\omega = u dx^I \in \Omega^k(U)$ and $\eta = v dx^J \in \Omega^l(U)$, where $u, v \in C^\infty(U)$.

-Claim: For any multi-index I, we have

$$d(u dx^I) = du \wedge dx^I.$$

Proof: If I has repeated indices, then $d(u dx^I) = 0 = du \wedge dx^I$. Otherwise, let σ be a permutation sending I to an increasing multi-index J. Then

$$d(u dx^I) = \text{sgn}(\sigma) d(u dx^J) = \text{sgn}(\sigma) du \wedge dx^J = du \wedge dx^I.$$

Using the claim, we compute:

$$\begin{aligned} d(\omega \wedge \eta) &= d((udx^I) \wedge (vdx^J)) \\ &= d(uv\, dx^I \wedge dx^J) \\ &= (vdu + udv) \wedge dx^I \wedge dx^J \\ &= (du \wedge dx^I) \wedge (vdx^J) + (-1)^k (udx^I) \wedge (dv \wedge dx^J) \\ &= d(u\, dx^I) \wedge (vdx^J) + (-1)^k (udx^I) \wedge d(vdx^J) \\ &= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta \end{aligned}$$

so we are done.

(c) We now deal with the case of a smooth 0-form u:

$$d(du) = d\left(\frac{\partial u}{\partial x^i} dx^i\right) = \frac{\partial^2 u}{\partial x^i \partial x^j} dx^i \wedge dx^j$$

$$= \sum_{i<j} \left(\frac{\partial^2 u}{\partial x^i \partial x^j} - \frac{\partial^2 u}{\partial x^j \partial x^i}\right) dx^i \wedge dx^j = 0.$$

as $dx^i \wedge dx^i = 0$. We now deal with the general case (i.e. $\omega \in \Omega^k(U)$):

$$\begin{aligned} d(d\omega) &= d\left(\sum_J d\omega_J dx^{j_1} \wedge \cdots \wedge dx^{j_k}\right) \\ &= \sum_J d(d\omega_J) dx^{j_1} \wedge \cdots \wedge dx^{j_k} + \sum_J (-1)^J\, d\omega_J \wedge d(dx^{j_1} \wedge \cdots \wedge dx^{j_k}) \\ &= 0 \end{aligned}$$

Thus, $d \circ d = 0$ holds for all forms.

(d) Due to (a), it suffices to consider $\omega = u\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, in which case we have

$$\begin{aligned} F^*(d(u\, dx^{i_1} \wedge \cdots \wedge dx^{i_k})) &= F^*(du \wedge\ dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \\ &= d(u \circ F) \wedge d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F)) \\ &= d\Big((u \circ F)\, d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F)\Big) \\ &= d\Big(F^*(u\, dx^{i_1} \wedge \cdots \wedge dx^{i_k})\Big) \end{aligned}$$

so we are done. □

Theorem 10.21 (Existence and uniqueness of exterior differentiation)**.** Let M be a smooth manifold. For each k, there are unique operators

$$d : \Omega^k(M) \to \Omega^{k+1}(M),$$

called <u>exterior differentiation</u>, satisfying the following properties:

(a) d is $\mathbb{R}$-linear.

(b) If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(c) $d \circ d \equiv 0$.

(d) For $f \in \Omega^0(M) = C^\infty(M)$, df is the differential of f, given by $df(X) = Xf$.

In any smooth chart, d is given by $(*_8)$.

Proof. - Existence: Given $\omega \in \Omega^k(U)$ for each smooth chart (U, φ) for M, we set $d\omega := \varphi^* d\left((\varphi^{-1})^*\omega\right)$. This is well-defined, since for any other smooth chart (V, ψ), the map $\varphi \circ \psi^{-1}$ is a diffeomorphism between open subsets of $\mathbb{R}^n$, so Proposition 10.20(d) yields

$$\begin{aligned} \psi^* d\left((\psi^{-1})^*\omega\right) &= (\varphi^{-1} \circ \varphi)^* \psi^* d\left((\psi^{-1})^*\omega\right) \\ &= (\varphi)^* \circ (\varphi^{-1})^* \circ \psi^* d\left((\psi^{-1})^*\omega\right) \\ &= (\varphi)^* d\left((\psi \circ \varphi^{-1})^* (\psi^{-1})^*\omega\right) \\ &= \varphi^* d\left((\varphi^{-1})^*\omega\right) \end{aligned}$$

as $F^* \circ G^* = (G \circ F)^*$. Moreover, d satisfies (a)-(d) by virtue of Proposition 10.20.

- Uniqueness: Suppose that d is any operator satisfying (a)-(d). We first show that d is determined locally: if ω_1 and ω_2 are k-forms that agree on an open subset $U \subseteq M$, then $d\omega_1 = d\omega_2$ on U. Let $p \in U$, set $\eta = \omega_1 - \omega_2$, and let $\psi \in C^\infty(M)$ be a bump function that is identically 1 on some neighborhood of p and supported in U. Then $\psi\eta$ is identically zero, so (a)-(d) imply that

$$0 = d(\psi\eta) = d\psi \wedge \eta + \psi d\eta.$$

Evaluating this at p and using that $\psi(p) = 1$ and $d\psi_p = 0$, we conclude that

$$0 = d\eta_p = d\omega_1|_p - d\omega_2|_p.$$

Now let $\omega \in \Omega^k(M)$ and let (U, φ) be a smooth chart on M. We write ω in coordinates as $\sum_I \omega_I dx^I$. For any $p \in U$, by means of a bump function we construct global smooth functions $\tilde{\omega}_I$ and $\tilde{x}^i$ on M that agree with ω_I and dx^i in a neighborhood of p. By virtue of (a)-(d), together with the observation in the previous paragraph, it follows that $(*_8)$ holds at p. Since p was arbitrary, this d must be equal to the one we defined above. $\square$

→ The differential on functions extends uniquely to an anti-derivation of $\Omega^*(M)$ of degree $+1$ whose square is zero.

Proposition 10.22 (Naturality of the exterior derivative)**.** If $F : M \to N$ is a smooth map, then for each k, the pullback map $F^* : \Omega^k(N) \to \Omega^k(M)$ commutes with d, i.e.,

$$F^*(d\omega) = d(F^*\omega), \quad \forall \omega \in \Omega^k(N).$$

Proof. We apply Proposition 10.20(d) to the coordinate representation $\psi \circ F \circ \varphi^{-1}$ of F, and on $U \cap F^{-1}(V)$, we obtain

$$\begin{aligned} F^*(d\omega) &= F^*\psi^* d((\psi^{-1})^*\omega) \\ &= \varphi^* \circ (\psi \circ F \circ \varphi^{-1})^* d\Big((\psi^{-1})^*\omega\Big) \\ &= \varphi^* d\Big((\psi \circ F \circ \varphi^{-1})^*(\psi^{-1})^*\omega\Big) \\ &= \varphi^* d\Big((\varphi^{-1})^* F^*\omega\Big) \\ &= d(F^*\omega). \end{aligned}$$

□

→ compute exterior derivatives of k-forms on $\mathbb{R}^3$.

Definition 10.23. Let M be a smooth manifold and let $\omega \in \Omega^k(M)$. We say that ω is closed if $d\omega = 0$, and exact if there exists $\eta \in \Omega^{k-1}(M)$ such that $\omega = d\eta$.

Remark 10.24. Every exact form is closed, since $d \circ d \equiv 0$, but the converse need not be true in general. However, it can be shown that closed forms are locally exact (but not necessarily globally). See also Poincaré lemma.

Chapter 11

Integration on Manifolds

11.1 Manifolds with boundary

We first give a crash course on manifolds with boundary. They play a central role in the theory of integration on manifolds, which will be briefly discussed afterwards; see pp. 135–144.

Definition 11.1. The closed n-dimensional upper half-space $\mathbb{H}^n \subseteq \mathbb{R}^n$ is defined as

$$\mathbb{H}^n = \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}.$$

The interior and the boundary of $\mathbb{H}^n$ as a subset of $\mathbb{R}^n$ are denoted by $\operatorname{Int} \mathbb{H}^n$ and $\partial \mathbb{H}^n$, respectively.

If $n > 0$, then

$$\begin{aligned} \operatorname{Int} \mathbb{H}^n &= \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n > 0\} \\ \partial \mathbb{H}^n &= \{(x^1, \dots, x^n) \in \mathbb{R}^n \mid x^n = 0\} \end{aligned}$$

whereas if $n = 0$, then

$$H^0 = \mathbb{R}^0 = \{0\}, \quad \partial H^0 = \emptyset.$$

Definition 11.2. An n-dimensional topological manifold with boundary is a second-countable Hausdorff space M in which every point has a neighborhood homeomorphic either to an open subset of $\mathbb{R}^n$ or to a (relatively) open subset of $\mathbb{H}^n$.

An open subset $U \subseteq M$ together with a map $\varphi : U \to \mathbb{R}^n$ that is a homeomorphism onto an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$ is called a chart for M. When it is necessary to make the distinction, we call (U, φ) an interior chart for M if $\varphi(U)$ is an open subset of $\mathbb{R}^n$ (which includes the case of an open subset of $\mathbb{H}^n$ that does not intersect with $\partial \mathbb{H}^n$) and a boundary chart if $\varphi(U)$ is an open subset of $\mathbb{H}^n$.

A point $p \in M$ is called an interior point of M if it is in the domain of some interior chart, and a boundary point of M if it is in the domain of a boundary chart that sends p to $\partial \mathbb{H}^n$. The set of all boundary points of M is denoted by ∂M and is called the boundary of M. The set of all interior points of M is denoted by $\text{Int} M$ and is called the interior of M.

Theorem 11.3 (Topological invariance of the boundary)**.** If M is a topological manifold with boundary, then each point of M is either a boundary point or an interior point, but not both. Thus, ∂M and $\text{Int}\, M$ are disjoint sets whose union is M.

Example 11.4.

1) Every interval in $\mathbb{R}$ is a (connected) 1-manifold with boundary, whose manifold boundary consists of its endpoints (if any).

2) The closed unit ball $\mathbb{B}^n \subseteq \mathbb{R}^n$ is an n-manifold with boundary, whose manifold boundary is $\mathbb{S}^{n-1}$.

Proposition 11.5. Let M be a topological n-manifold with boundary.

1. $\text{Int}\, M$ is an open subset of M and a topological n-manifold without boundary.

2. ∂M is a closed subset of M and a topological $(n-1)$-manifold without boundary.

3. M is a topological manifold (in the sense of Definition 2.1) if and only if $\partial M = \emptyset$.

Proof. Exercise! □

Next, if U is an open subset of $\mathbb{H}^n$, then a map $F : U \to \mathbb{R}^k$ is said to be smooth if for each $x \in U$, there exist an open subset $\tilde{U} \subseteq \mathbb{R}^n$ containing x and a smooth map $\tilde{F} : \tilde{U} \to \mathbb{R}^k$ that agrees with F on $\tilde{U} \cap U$. If F is such a map, then the restriction of F to $U \cap \text{Int}\, \mathbb{H}^n$ is smooth in the usual sense. By continuity, all partial derivatives of F at points of $U \cap \text{Int}\, \mathbb{H}^n$ are determined by their values in $\text{Int}\, \mathbb{H}^n$, and thus in particular are independent of the choice of extension.

Definition 11.6. Let M be a topological manifold with boundary. A smooth structure for M is defined to be a maximal smooth atlas (a collection of charts whose domains cover M and whose transition maps and their inverses are smooth in the sense just described). With such a structure, M is called a smooth manifold with boundary.

In the following remark, we collect some basic definitions and facts about smooth manifolds with boundary.

Remark 11.7. 1) Cf Chapter 3: smoothness of a map $F : M \to N$ between manifolds with boundary is defined in the same way (see Definition 3.3), with the usual understanding that a map whose domain is a subset of $\mathbb{H}^n$ is smooth if it admits an extension to a smooth map in a neighborhood of each point, and a map whose codomain is a subset of $\mathbb{H}^n$ is smooth if it is smooth as a map into $\mathbb{R}^n$. Smooth partitions of unity exist on smooth manifolds with boundary.

2) Cf Chapter 5: If M is a smooth n-manifold with boundary, then the tangent space T_pM to M at $p \in M$ is defined in the same way (see Definition 5.4), and it is an n-dimensional real vector space. For any smooth chart (U, φ) containing p, the coordinate vectors

$$\frac{\partial}{\partial x^1}\bigg|_p, \dots, \frac{\partial}{\partial x^n}\bigg|_p$$

(where $\frac{\partial}{\partial x^n}\big|_p$ should be interpreted as a one-sided derivative when $p \in \partial M$) form a basis for T_pM.

The differential of a smooth map $F : M \to N$ between smooth manifolds with boundary is defined in the same way (see Definition 5.6) and has the same representation in coordinates bases.

3) Cf Chapter 6: Submersions, immersions, embeddings, and local diffeomorphisms are defined in the same way (see Definition 6.2), and there is a version of the rank theorem in this setting; see [1], Theorem 4.15 and Problem 4.3.

4) Cf Chapter 7: Immersed and embedded submanifolds of smooth manifolds with boundary are defined in the same way (see Definitions 7.1 and 7.9) and are themselves smooth manifolds with (possibly empty) boundary.

$\to$ Properties of embedded submanifold with boundary = [1], Proposition 5.49.

$\to$ Version of the regular level set theorem in this setting = [1], Problem 5.23.

Theorem. If M is a smooth n-manifold with boundary, then with the subspace topology, ∂M is a topological $(n-1)$-manifold (without boundary), and has a unique smooth structure such that it is a properly embedded submanifold of M.

5) Cf Chapter 9: The tangent bundle of a smooth n-manifold with boundary is defined in the same way (see Definition 5.6), and it is a smooth vector bundle of rank n over the given manifold (see Proposition 8.5). Vector fields are also defined in the same way (see Definition 9.1), but flows need to be treated with extra care; see [1], Subsection 9.4.

Let M be a smooth manifold with boundary and let $p \in \partial M$. It is intuitively evident that the vectors in T_pM can be separated in three classes: those tangent to the boundary, those pointing inward, and those pointing outward. Formally, we make the following.

Definition. If $p \in \partial M$, then a vector $v \in T_pM \backslash T_p\partial M$ is said to be inward-pointing if for some $\epsilon > 0$ there exists a smooth curve $\gamma : [0, \epsilon) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$, and it is outward-pointing if there exists such a curve with domain $(-\epsilon, 0]$.

Proposition. Let M be a smooth n-manifold with boundary, $p \in \partial M$, and (x^i) be any smooth boundary coordinates defined on a neighborhood of p. The inward-pointing vectors in T_pM are precisely those with positive x^n-component, the outward-pointing ones are those with negative x^n-component, and the ones tangent to ∂M are those with zero x^n-component. Thus, T_pM is the disjoint union of $T_p\partial M$, the set of inward-pointing vectors, and the set of outward-pointing vectors. Finally, $v \in T_pM$ is inward-pointing if and only if $-v$ is outward-pointing.

The next result is used when discussing "boundary orientations"; see [Orientations, Subsection 2.3].

Proposition If M is a smooth manifold with boundary, then there exists a global smooth vector field on M whose restriction to ∂M is everywhere inward-pointing, and one whose restriction to ∂M is everywhere outward-pointing.

6) Cf Chapter 10: The cotangent bundle T^*M (respectively the k-th exterior power $\Lambda^k(T^*M)$ of the cotangent of a smooth n-manifold with boundary is defined in the same way (see Definition 10.2, respectively Definition 10.15(b)) and it is a smooth vector bundle of rank n (resp. of rank $\binom{n}{k}$) over M (see Proposition 10.3, resp p118 of [1]). Differential k-forms ($0 \le k \le n$) are also defined in the same way (see Definition 10.15(b)), and so does their exterior derivative as well (see Theorem 10.21).

Orientations of manifolds, which also play an important role in integration theory on manifolds, are briefly discussed in the PDF [Orientations].

11.2 Integral of a compactly supported form

We are now ready to develop the general theory of integration on oriented manifolds. We first define the integral of a differential form over a domain in Euclidean space, and then we show how to use diffeomorphism invariance and smooth partitions of unity to extend this definition to n-forms on oriented n-manifolds. The key feature of this definition is that it is invariant under orientation-preserving diffeomorphisms. Afterwards, we state Stokes' Theorem (without proof), which is a generalization of the fundamental theorem of calculus, and we also provide some applications; see pp. 142-144.

Definition 11.8. Let U be an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$ and let ω be a compactly supported n-form on U. We define

$$\int_U \omega := \int_D \omega,$$

where $D \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$ is any domain of integration (e.g., a rectangle) containing $\operatorname{supp}\omega$, and ω is extended to be zero on the complement of its support.

Note that Definition 11.8 does not depend on the choice of domain of integration (which is a bounded subset of $\mathbb{R}^n$ whose boundary has measure zero). Moreover, since ω can be written as $\omega = f\,dx^1 \wedge \cdots \wedge dx^n$ for some continuous function $f : \operatorname{supp}\omega \to \mathbb{R}$, the right-hand side in the above definition is (the usual integral)

$$\int_D \omega = \int_D f\,dx^1 \dots dx^n$$

Proposition 11.9. Let D and E be open domains of integration in $\mathbb{R}^n$ or $\mathbb{H}^n$, and let $G : \overline{D} \to \overline{E}$ be a smooth map that restricts to an orientation-preserving or orientation-reversing diffeomorphism $D \to E$. If ω is an n-form on $\overline{E}$, then

$$\int_D G^*\omega = \begin{cases} \int_E \omega, & \text{if } G \text{ is orientation preserving,} \\ -\int_E \omega, & \text{if } G \text{ is orientation reversing.} \end{cases}$$

Proof. Follows from the (usual) change of variables formula and the pullback formula for n-forms (Proposition 10.18). □

Since we cannot guarantee that arbitrary open or compact subsets are domains of integration, we need the following lemma in order to extend Proposition 11.9 to compactly supported n-forms defined on open subsets; see Proposition 11.11.

Lemma 11.10. Let U be an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$ and let K be a compact subset of U. There is an open domain of integration D such that

$$K \subseteq D \subseteq \overline{D} \subseteq U.$$

Proposition 11.11. Let U and V be open subsets of $\mathbb{R}^n$ or $\mathbb{H}^n$, and $G : U \to V$ be an orientation-preserving or orientation-reversing diffeomorphism. If ω is a compactly supported n-form on V, then:

$$\int_V \omega = \begin{cases} \int_U G^*\omega, & \text{if } G \text{ is orientation preserving} \\ -\int_U G^*\omega, & \text{if } G \text{ is orientation reversing} \end{cases}$$

Proof. By Lemma 11.10, there is an open domain of integration E such that $\operatorname{supp}\omega \subseteq E \subseteq \overline{E} \subseteq V$. Since diffeomorphisms take interiors to interiors, boundaries to boundaries, and sets of measure zero to sets of measure zero, $D = G^{-1}(E) \subseteq U$ is an open domain of integration containing $\operatorname{supp} G^*\omega$. We conclude by Proposition 11.9. □

11.3 Integration over the whole manifold

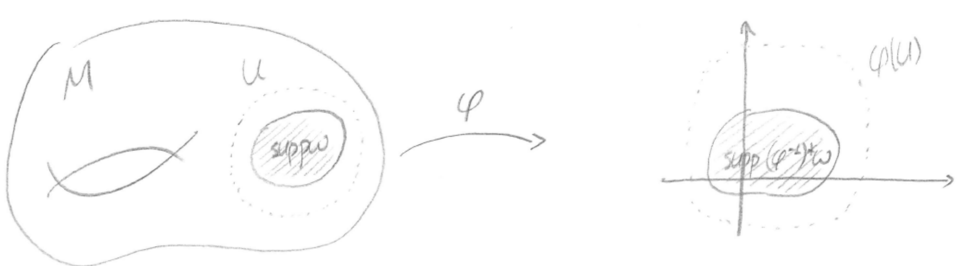
Using the above proposition, we can now make sense of the integral of a differential form over an oriented manifold.

Let M be an oriented smooth n-manifold with or without boundary, and let ω be an n-form on M. Suppose first that ω is compactly supported in the domain of a single smooth chart (U, φ) that is either positively or negatively oriented.

We define the integral of ω over M to be:

$$\int_M \omega := \pm \int_{\varphi(U)} (\varphi^{-1})^* \omega \tag{$*_1$}$$

with the positive sign for a positively oriented chart, and the negative sign otherwise. Since $(\varphi^{-1})^*\omega$ is a compactly supported n-form on the open subset $\varphi(U) \subseteq \mathbb{R}^n$ or $\mathbb{H}^n$, its integral is defined as in Definition 11.8, see pp. 135–136.



Proposition 11.12. If M and ω are as above, then $\int_M \omega$ does not depend on the choice of smooth chart whose domain contains $\operatorname{supp} \omega$.

Proof. Let (U, φ) and $(\tilde{U}, \tilde{\varphi})$ be two smooth charts such that $\operatorname{supp} \omega \subseteq U \cap \tilde{U}$. If both charts are similarly oriented, then $\tilde{\varphi} \circ \varphi^{-1} : \varphi(U \cap \tilde{U}) \to \tilde{\varphi}(U \cap \tilde{U})$ is an orientation-preserving diffeomorphism, see [Orientations, Ex. 25], so Proposition 11.11 yields:

$$\begin{aligned} \int_{\tilde{\varphi}(\tilde{U})} (\tilde{\varphi}^{-1})^* \omega &= \int_{\tilde{\varphi}(U \cap \tilde{U})} (\tilde{\varphi}^{-1})^* \omega = \int_{\varphi(U \cap \tilde{U})} (\tilde{\varphi} \circ \varphi^{-1})^* (\tilde{\varphi}^{-1})^* \omega \\ &= \int_{\varphi(U \cap \tilde{U})} (\varphi^{-1})^* \underbrace{(\tilde{\varphi})^* (\tilde{\varphi}^{-1})^*}_{(\tilde{\varphi}^{-1} \circ \tilde{\varphi})^* = \mathrm{Id}^*} \omega = \int_{\varphi(U)} (\varphi^{-1})^* \omega \end{aligned}$$

If the charts are oppositely oriented, then the two diffs given by $(*_1)$ have opposite signs, but this is compensated by the fact that $\tilde{\varphi} \circ \varphi^{-1}$ is orientation-reversing, so Proposition 11.11 introduces an extra negative sign into the above computation. □

Let M be as above and let ω be a compactly supported n-form on M. Let $\{U_i\}$ be a finite open cover of $\operatorname{supp} \omega$ by domains of positively or negatively oriented smooth charts, and let $\{\psi_i\}$ be a smooth partition of unity subordinate to this covering. We define the integral of ω over M to be

$$\int_M \omega := \sum_i \int_M \psi_i \omega \tag{$*_2$}$$

Since for each i, the n-form $\psi_i\omega$ is compactly supported in U_i, each of the terms in this (finite) sum is well defined according to our previous discussion. The following proposition shows that the integral is well defined.

Proposition 11.13. The definition ($*_2$) does not depend on the choice of open cover or partition of unity.

Proof. Let $\{\tilde{U}_j\}$ be another finite open cover of $\operatorname{supp}\omega$ by domains of positively or negatively oriented smooth charts, and let $\{\tilde{\psi}_j\}$ be a subordinate smooth partition of unity. Since

$$\int_M \psi_i\omega = \int_M \left(\sum_j \tilde{\psi}_j\right) \psi_i\omega = \sum_j \int_M \tilde{\psi}_j\psi_i\omega, \quad \forall i$$

we obtain

$$\sum_i \int_M \psi_i\omega = \sum_{i,j} \int_M \tilde{\psi}_j\psi_i\omega$$

Each term in this sum is the integral of a form that is compactly supported in the domain of a single smooth chart (e.g. in U_i), so by Proposition 11.12, each term is well defined, regardless of which coordinate map we use to compute it. The same argument, starting with $\int_M \tilde{\psi}_j\omega$ instead, shows that:

$$\sum_j \int_M \tilde{\psi}_j\omega = \sum_{i,j} \int_M \tilde{\psi}_j\psi_i\omega.$$

Thus, both definitions yield the same value for $\int_M \omega$. □

If $S \subseteq M$ is an oriented immersed k-dimensional submanifold (with or without boundary) and ω is a k-form on M whose restriction to S is compactly supported, then we interpret $\int_S \omega$ as $\int_S i^*\omega$, where $i : S \to M$ is the inclusion map. In particular, if M is a compact, oriented, smooth n-manifold with boundary and ω is an $(n-1)$-form on M, then we can interpret $\int_{\partial M} \omega$ unambiguously as the integral of $i^*_{\partial M}\omega$ over ∂M, where ∂M is always understood to have the induced orientation; see [Orientations, Prop. 22].

Proposition 11.14 (Properties of integrals)**.** Let M and N be non-empty oriented smooth n-manifolds with or without boundary, and let ω and η be compactly supported n-forms on M. Then:

(a) Linearity: If $a, b \in \mathbb{R}$, then:

$$\int_M a\omega + b\eta = a\int_M \omega + b\int_M \eta.$$

(b) Orientation reversal: If $-M$ denotes M with the opposite orientation, then:

$$\int_{-M} \omega = -\int_M \omega.$$

(c) Positivity: If ω is a positively oriented orientation form, then:

$$\int_M \omega > 0.$$

(d) Diffeomorphism invariance: If $F : N \to M$ is an orientation-preserving or orientation-reversing diffeomorphism, then:

$$\int_M \omega = \begin{cases} \int_N F^*\omega, & \text{if } F \text{ is orientation preserving,} \\ -\int_N F^*\omega, & \text{if } F \text{ is orientation reversing.} \end{cases}$$

Proof. (a) Exercise.

(b) Exercise (follows from the usual change of variables formula).

(c) Since ω is a positively oriented orientation form on M, if (U, φ) is a positively oriented smooth chart, then $(\varphi^{-1})^*\omega$ is a positive function times $dx^1 \wedge \cdots \wedge dx^n$ (while if (U, φ) is negatively oriented, then it is a negative function times the same form); see the proof of [Orientations, Proposition 14]. Therefore, each term in $(*_2)$ defining $\int_M \omega$ is non-negative, with at least one strictly positive term; this proves (c).

(d) It suffices to treat the case when ω is compactly supported in a single positively or negatively oriented smooth chart. If (U, φ) is a positively oriented such chart and if F is orientation-preserving, then it is easy to check that $(F^{-1}(U), \varphi \circ F)$ is an oriented smooth chart on N whose domain contains supp $F^*\omega$, so the result follows from Proposition 11.11. The remaining cases follow from this one and (b).

□

11.4 Stokes Theorem and Applications

We now state (without proof) the central result in the theory of integration on manifolds, Stokes' theorem. It is a far-reaching generalization of the fundamental theorem of calculus and of the classical theorems of vector calculus.

Theorem 11.15 (Stokes' Theorem)**.** Let M be an oriented smooth n-manifold with boundary, and let ω be a compactly supported smooth $(n-1)$-form on M. Then:

$$\int_M d\omega = \int_{\partial M} \omega.$$

Here, ∂M is understood to have the induced (Stokes) orientation, and the ω on the right-hand side is to be interpreted as $\iota_{\partial M}^*\omega$. If $\partial M = \varnothing$, then the right-hand side is to be interpreted as 0. When M is 1-dimensional, the right-hand side is just a finite sum.

Finally, let us see some applications of Stokes' theorem.

Example 11.16. Let M be a smooth manifold. Let $\gamma : [a,b] \to M$ be a smooth embedding, so that $S := \gamma([a,b])$ is an embedded 1-submanifold with boundary in M. If we give S the orientation such that γ is orientation-preserving, then for any $f \in C^\infty(M)$, Stokes' theorem says that:

$$\int_\gamma df = \int_{[a,b]} \gamma^* df = \int_S df = \int_{\partial S} f = f(\gamma(b)) - f(\gamma(a)).$$

(because $\partial S = \{\gamma(a), \gamma(b)\}$ is 0-dimensional). In particular, when $\gamma : [a,b] \to \mathbb{R}$ is the inclusion map, then Stokes' theorem is just the ordinary fundamental theorem of calculus.

Theorem 11.17 (Green's Theorem)**.** Let $D \subseteq \mathbb{R}^2$ be a compact regular domain (i.e., a properly embedded codimension-0 submanifold with boundary), and let P, Q be smooth real-valued functions on D. Then:

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \int_{\partial D} Pdx + Qdy.$$

Proof. Apply Stokes' theorem to the 1-form $Pdx + Qdy$. □

Corollary 11.18 (Integrals of exact forms)**.** If M is a compact, oriented, smooth n-manifold without boundary, then the integral of every exact n-form over M is zero:

$$\int_M d\omega = 0.$$

Corollary 11.19 (Integrals of closed forms over boundaries)**.** Let M be a compact, oriented, smooth n-manifold with boundary. If ω is a closed $(n-1)$-form on M, then the integral of ω over ∂M is zero

$$\int_{\partial M} \omega = 0 \quad \text{if } d\omega = 0 \text{ on } M$$

Corollary 11.20. Let M be a smooth manifold with or without boundary, let $S \subseteq M$ be an oriented, compact, smooth k-dimensional submanifold (without boundary), and let ω be a closed k-form on M. If $\int_S \omega \neq 0$, then both of the following are true:

(a) ω is not exact on M.

(b) S is not the boundary of an oriented, compact, smooth submanifold with boundary in M.

Proof. (a) If ω were exact on M, then $\omega = d\eta$ for some $(k-1)$-form on M, so:

$$0 \neq \int_S \omega = \int_S i_S^* \omega = \int_S i_S^* d\eta = \int_S d(i_S^* \eta) = 0$$

which is a contradiction.

(b) Argue again by contradiction and invoke (11.19).

$\square$

Chapter 12

Multilinear Algebra

The following two documents contain the notes on Multilinear Algebra (relevant for n-forms and tensors) and Orientation (relevant for integration on manifolds) and were written by Dr. Tsakanikas and Linus Rösler during the Fall semester of 2023-2024. Typos may be present and, just as before, these topics are presented in various books (like [1] for example) which you should refer to if you have any doubts.

12.1 The Dual of a Vector Space

Definition 12.1. Let V be a finite-dimensional real vector space.

(a) A covector on V is a real-valued linear functional on V, i.e., a linear map $\omega \colon V \to \mathbb{R}$.

(b) The set of all covectors on V is a real vector space under the obvious operations of pointwise addition and scalar multiplication. It is denoted by V^* and is called the dual space of V.

The next proposition expresses the most important fact about V^*.

Proposition 12.2. Let V be a real vector space of dimension n. Given any basis $(E_1, \dots, E_n)$ for V, consider the covectors $\varepsilon^1, \dots, \varepsilon^n \in V^*$ defined by

$$\varepsilon^i(E_j) = \delta^i_j.$$

Then $(\varepsilon^1, \dots, \varepsilon^n)$ is a basis for V^*, called the dual basis to (E_j). In particular,

$$\dim_{\mathbb{R}} V = \dim_{\mathbb{R}} V^*.$$

Proof. Exercise! □

In general, if (E_j) is a basis for V and if (ε^i) is its dual basis, then for any vector $v = v^j E_j \in V$ we have

$$\varepsilon^i(v) = v^j \varepsilon^i(E_j) = v^j \delta^i_j = v^i.$$

Thus, the i-th basis covector ε^i picks out the i-th component of a vector with respect to the basis (E_j).

More generally, we can express an arbitrary covector $\omega \in V^*$ in terms of the dual basis as

$$\omega = \omega_i \varepsilon^i,$$

where the i-th component is determined by $\omega_i = \omega(E_i)$. Thus, the action of the given covector $\omega \in V^*$ on a vector $v = v^j E_j \in V$ is

$$\omega(v) = \omega_i v^j \varepsilon^i(E_j) = \omega_i v^i.$$

Let V and W be real vector spaces and let $A\colon V \to W$ be a linear map. The dual map of A is the linear map $A^*\colon W^* \to V^*$ defined by

$$(A^*\omega)(v) := \omega(Av),\ \ \omega \in W^*,\ \ v \in V.$$

It is straightforward to check that it satisfies the following properties:

(a) $(A \circ B)^* = B^* \circ A^*$.

(b) $(\mathrm{Id}_V)^* = \mathrm{Id}_{V^*}$.

Proposition 12.3. The assignment that sends a vector space to its dual space and a linear map to its dual linear map is a contravariant functor from the category of real vector spaces to itself.

Another important fact about the dual of a finite-dimensional vector space is the following.

Proposition 12.4. Let V be a finite-dimensional real vector space. For any given $v \in V$, define a linear functional $\xi(v)$ by

$$\begin{aligned} \xi(v)\colon V^* &\to \mathbb{R} \\ \omega &\mapsto \xi(v)(\omega) := \omega(v). \end{aligned}$$

Then $\xi(v) \in (V^*)^*$, that is, $\xi(v)$ is a linear functional on V^*. Moreover, the map

$$\begin{aligned} \xi\colon V &\to (V^*)^* \\ v &\mapsto \xi(v) \end{aligned}$$

is an $\mathbb{R}$-linear isomorphism, which is canonical (it is defined without reference to any basis).

Proof. The proof that both $\xi(v)$ and ξ are linear maps are left as exercises. Since by Proposition 12.2 we have

$$\dim V = \dim V^* = \dim(V^*)^*,$$

it suffices to prove that ξ is injective. To this end, let $v \in V$ be non-zero, complete it to a basis $v = E_1, E_2, \ldots, E_n$ of V, and let (ε^i) be its dual basis. Then

$$\xi(v)(\varepsilon^1) = \varepsilon^1(v) = \varepsilon^1(E_1) = 1,$$

so $\xi(v) \neq 0$. Therefore, $\ker \xi = 0$; in other words, ξ is injective, as desired. □

Due to Proposition 12.4, the real number $\omega(v)$ obtained by applying a covector ω to a vector v is sometimes denoted by either of the more symmetric-looking notations $\langle \omega, v \rangle$ or $\langle v, \omega \rangle$; both expressions can be thought of either as the action of the covector $\omega \in V^*$ on the vector $v \in V$, or as the action of the linear functional $\xi(v) \in V^{**}$ on the element $\omega \in V^*$. There should be no cause for confusion with the use of the same angle bracket notation for inner products: whenever one of the arguments is a vector and the other a covector, the notation $\langle \omega, v \rangle$ is always to be interpreted as the natural pairing between vectors and covectors, not as an inner product.

There is also a symmetry between bases and dual bases for a finite-dimensional vector space V: any basis for V determines a dual basis for V^*, and conversely, any basis for V^* determines a dual basis for $V^{**} = V$. If (ε^i) is the basis for V^* dual to a basis (E_j) for V, then (E_j) is the basis dual to (ε^i), because both statements are equivalent to the relation $\langle \varepsilon^i, E_j \rangle = \delta^i_j$.

12.2 Multilinear Maps and Tensors

In the preceding section, we defined and briefly examined the dual of a vector space (in the finite-dimensional case), which is the space of real-valued linear functions on the given vector space. A natural, and from the point of view of (differential) geometry very important, generalization is to consider functions with several arguments, which are linear in each individual argument. These are called multilinear functions.

Definition 12.5. Let $V_1, \ldots, V_k$ and W be real vector spaces. A map $F\colon V_1 \times \cdots \times V_k \to W$ is called multilinear if it is linear as a function of each variable separately when the others are held fixed; that is, if $1 \leq i \leq k$ is arbitrary, and if we are given elements $v_i, v_i' \in V_i$ and real numbers $a, a' \in \mathbb{R}$, then

$$F(v_1, \ldots, av_i + a'v_i', \ldots, v_k) = aF(v_1, \ldots, v_i, \ldots, v_k) + a'F(v_1, \ldots, v_i', \ldots, v_k).$$

Denote by $L(V_1, \ldots, V_k; W)$ the set of multilinear maps from $V_1 \times \cdots \times V_k$ to W, and note that $L(V_1, \ldots, V_k; W)$ has the structure of a real vector space. In the special case when $V_1 = \ldots = V_k = V$ and $W = \mathbb{R}$, we often call an element of the space $L(V, \ldots, V; \mathbb{R})$ a k-multilinear function on V; see Definition 12.7.

Now, if the target space is $W = \mathbb{R}$, then there is a simple operation with which one can successively build multilinear maps.

Definition 12.6. Let $V_1, \dots, V_k$ and $W_1, \dots, W_l$ be real vector spaces, and consider $F \in L(V_1, \dots, V_k; \mathbb{R})$ and $G \in L(W_1, \dots, W_l; \mathbb{R})$. The function

$$F \otimes G \colon V_1 \times \cdots \times V_k \times W_1 \times \cdots \times W_l \to \mathbb{R}$$
$$(v_1, \dots, v_k, w_1, \dots, w_l) \mapsto F(v_1, \dots, v_k)\, G(w_1, \dots, w_l)$$

is called the tensor product of F and G.

Exercise 12.7. (a) Show that, given F and G as above, the function $F \otimes G$ is multilinear, that is,

$$F \otimes G \in L(V_1, \dots, V_k, W_1, \dots, W_l; \mathbb{R}).$$

(b) Show that the tensor product operation

$$- \otimes - \colon L(V_1, \dots, V_k; \mathbb{R}) \times L(W_1, \dots, W_l; \mathbb{R}) \to L(V_1, \dots, V_k, W_1, \dots, W_l; \mathbb{R})$$
$$(F, G) \mapsto F \otimes G$$

is bilinear, i.e., multilinear with two variables, and associative, i.e., for any multilinear real-valued functions F, G, H, we have $F \otimes (G \otimes H) = (F \otimes G) \otimes H$.

Given a finite-dimensional real vector space V, we described in section 12.1 how to obtain a basis for the dual space $V^* = L(V; \mathbb{R})$ from a basis for V. With the above operation at hand, we may now generalize this to the space $L(V_1, \dots, V_k; \mathbb{R})$.

Proposition 12.8. Let $V_1, \dots, V_k$ be $\mathbb{R}$-vector spaces of dimensions $n_1, \dots n_k$, respectively. For each $1 \leq j \leq k$, let $(E_1^{(j)}, \dots, E_{n_j}^{(j)})$ be a basis of V_j, and denote by $(\varepsilon^1_{(j)}, \dots, \varepsilon^{n_j}_{(j)})$ the corresponding dual basis of V_j^*. Then the set

$$\mathcal{B} := \left\{ \varepsilon^{i_1}_{(1)} \otimes \cdots \otimes \varepsilon^{i_k}_{(k)} \mid 1 \leq i_1 \leq n_1, \dots, 1 \leq i_k \leq n_k \right\}$$

is a basis for $L(V_1, \dots, V_k; \mathbb{R})$, which therefore has dimension $n_1 \dots n_k$.

Proof. First, given $F \in L(V_1, \dots, V_k; \mathbb{R})$, define for each multi-index $I = (i_1, \dots, i_k)$ with $1 \leq i_j \leq n_j$ for all $1 \leq j \leq k$, a number $F_I \in \mathbb{R}$ by

$$F_I := F\left(E^{(1)}_{i_1}, \dots, E^{(k)}_{i_k}\right).$$

Also, use the short-hand notation

$$\varepsilon^{\otimes I} := \varepsilon^{i_1}_{(1)} \otimes \cdots \otimes \varepsilon^{i_k}_{(k)}.$$

We will show that

$$F = \sum_I F_I\, \varepsilon^{\otimes I},$$

where the sum is taken over all multi-indices as above, and thereby show that $\mathcal{B}$ spans $L(V_1, \dots, V_k; \mathbb{R})$.

To this end, take $(v_1, \dots, v_k) \in V_1 \times \cdots \times V_k$. For integers i_j between 1 and n_j, let $v_j^{i_j} \in \mathbb{R}$ be the coefficient of v_j with respect to the basis $(E_1^{(j)}, \dots, E_{n_j}^{(j)})$, i.e.,

$$v_j^{i_j} = \varepsilon_{(j)}^{i_j}(v_j).$$

Then by the multilinearity of F we have

$$F(v_1, \dots, v_k) = \sum_I v_1^{i_1} \cdots v_k^{i_k} F\left(E_{i_1}^{(1)}, \dots, E_{i_k}^{(k)}\right) = \sum_I v_1^{i_1} \cdots v_k^{i_k} F_I.$$

On the other hand, we have

$$\left[\sum_I F_I\, \varepsilon^{\otimes I}\right](v_1, \dots, v_k) = \sum_I F_I\, \varepsilon^{\otimes I}(v_1, \dots, v_k) = \sum_I v_1^{i_1} \cdots v_k^{i_k} F_I.$$

Hence F and $\sum_I F_I \varepsilon^{\otimes I}$ agree at any k-tuple and thus are equal, so $\mathcal{B}$ indeed spans $L(V_1, \dots, V_k; \mathbb{R})$.

Finally, in order to see that $\mathcal{B}$ is linearly independent, suppose that we have

$$\sum_I \lambda_I\, \varepsilon^{\otimes I} = 0$$

for some real numbers $\lambda_I \in \mathbb{R}$ indexed by multi-indices I. Evaluating both sides at $\left(E_{i_1}^{(1)}, \dots, E_{i_k}^{(k)}\right)$ for some fixed multi-index $I = (i_1, \dots, i_k)$, we obtain by the same computation as above that $\lambda_I = 0$. Hence, $\mathcal{B}$ is linearly independent. □

The proof of Proposition 12.8 shows also that the components $F_{i_1 \dots i_k}$ of a multilinear function F in terms of the basis elements in $\mathcal{B}$ are given by

$$F_{i_1 \dots i_k} = F\left(E_{i_1}^{(1)}, \dots, E_{i_k}^{(k)}\right).$$

Thus, F is completely determined by its action on all possible sequences of basis vectors.

Remark 12.9. You might have already encountered the abstract construction of the tensor product of vector space If so, then regarding the above discussion (which shows that the real vector space $L(V_1, \dots, V_k; \mathbb{R})$ can be viewed as the set of all linear combinations of objects of the form $\omega^1 \otimes \cdots \otimes \omega^k$, where $\omega^i \in V_i^*$ are covectors), one should remark the following: given finite-dimensional real vector spaces $V_1, \dots, V_k$, there is a canonical isomorphism

$$V_1^* \otimes \cdots \otimes V_k^* \cong L(V_1, \dots, V_k; \mathbb{R}),$$

which is induced by the multilinear map

$$\begin{aligned}\Phi \colon V_1^* \times \ldots \times V_k^* &\to L(V_1, \ldots, V_k; \mathbb{R}) \\ \Phi\left(\omega^1, \ldots, \omega^k\right)(v_1, \ldots, v_k) &\coloneqq \left(\omega^1 \otimes \cdots \otimes \omega^k\right)(v_1, \ldots, v_k) \\ &= \omega^1(v_1) \cdots \omega^k(v_k).\end{aligned}$$

Under this canonical isomorphism, abstract tensors correspond to the concrete tensor product of multilinear functions defined above. As it is a natural isomorphism, we may use the expression $V_1^* \otimes \cdots \otimes V_k^*$ as a notation for $L(V_1, \ldots, V_k; \mathbb{R})$ (this is a typical example of slight abuse of notation, where one identifies naturally isomorphic objects). Finally, using Proposition 12.4, we also obtain a canonical identification

$$V_1 \otimes \cdots \otimes V_k \cong L(V_1^*, \ldots, V_k^*; \mathbb{R}).$$

Therefore, we may view the above construction as a concrete construction of the abstract tensor product.

Let us now turn our attention to various spaces of multilinear functions on a finite-dimensional real vector space that naturally appear in (differential) geometry.

Definition 12.10. Let V be a finite-dimensional real vector space. For any integer $k \geq 1$, we denote by $T^k(V^*)$ the space of k-multilinear functions on V, i.e.,

$$T^k(V^*) \coloneqq L(\underbrace{V, \ldots, V}_{k \text{ times}}; \mathbb{R}) \cong \underbrace{V^* \otimes \ldots \otimes V^*}_{k \text{ copies}}.$$

By convention, we also define $T^0(V^*) \coloneqq \mathbb{R}$. The elements of $T^k(V^*)$ are often referred to as covariant k-tensors on V.

Observe that every linear functional $\omega \colon V \to \mathbb{R}$ is (trivially) multilinear, so a covariant 1-tensor is just a covector on V. Thus,

$$T^1(V^*) = V^*.$$

According to Proposition 12.8, we obtain a basis for $T^k(V^*)$ as follows. Assume that V has dimension n, let $(E_1, \ldots, E_n)$ be a basis for V and denote by $(\varepsilon^1, \ldots, \varepsilon^n)$ the dual basis for V^*. For a multi-index $I = (i_1, \ldots, i_k)$, where $1 \leq i_j \leq n$ for all j, define the elementary covariant k-tensor $\varepsilon^{\otimes I}$ by the formula

$$\varepsilon^{\otimes I} \coloneqq \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k}$$

(see the proof of Proposition 12.8)and for an integer $m \in \mathbb{Z}_{\geq 1}$, denote by $[m]$ the set $\{1, \ldots, m\}$. Then the set

$$\left\{\varepsilon^{\otimes I} \mid I \in [n]^{[k]}\right\}$$

is a basis for $T^k(V^*)$; in particular, we have

$$\dim_{\mathbb{R}} T^k(V^*) = n^k.$$

Therefore, every covariant k-tensor $\alpha \in T^k(V^*)$ can be written uniquely in the form

$$\alpha = \alpha_I \, \varepsilon^{\otimes I} = \alpha_{i_1 \dots i_k} \, \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k},$$

where the n^k coefficients $\alpha_I = \alpha_{i_1 \dots i_k}$ are determined by

$$\alpha_{i_1 \dots i_k} = \alpha(E_{i_1}, \dots, E_{i_k}).$$

For example, $T^2(V^*)$ is the space of bilinear forms on V – note that a covariant 2-tensor on V is simply a real-valued bilinear function of two vectors – and every bilinear form on V can be written as $\beta = \beta_{ij} \, \varepsilon^i \otimes \varepsilon^j$ for some uniquely determined $n \times n$ matrix (β_{ij}).

Definition 12.11. For a covariant k-tensor $\alpha \in T^k(V^*)$ and a permutation $\sigma \in S_k$, denote by ${}^\sigma\alpha$ the covariant k-tensor given by

$$\begin{aligned} {}^\sigma\alpha \colon V \times \cdots \times V &\to \mathbb{R} \\ (v_1, \dots, v_k) &\mapsto \alpha\big(v_{\sigma(1)}, \dots, v_{\sigma(k)}\big). \end{aligned}$$

In the following two sections we will discuss two important subspaces of $T^k(V^*)$, namely the subspaces of symmetric resp. alternating covariant k-tensors. Both are described by the way that a permutation of the arguments of the given covariant k-tensor changes its value. A significant application of symmetric tensors in the theory of smooth manifolds is in the form of Riemannian metrics. Loosely speaking, a Riemannian metric is a choice of an inner product on each tangent space of the given manifold, varying smoothly from point to point, and allows one to define geometric concepts such as lenghts, angles and distances on the manifold. Riemannian metrics will not be discussed in this course, and this is the main reason why the discussion about symmetric tensors in section 12.3 will be kept to a minimum. On the other hand, differential forms will be discussed thoroughly in Lecture 13 and Lecture 14 of this course. They constitute a significant application of alternating tensors in smooth manifold theory, and they will be presented in section 12.4.

12.3 Symmetric Tensors

In all probability, you have already encountered the concept of inner product on a finite-dimensional real vector space V. It is a bilinear map $\langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{R}$ which is symmetric and positive definite; in particular, $\langle \cdot, \cdot \rangle$ is a covariant 2-tensor on V, having the additional property that its value is unchanged when the two input arguments are exchanged; namely, we have $\langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle$ for any $v_1, v_2 \in V$. We now generalize this notion to any covariant k-tensor on V.

Definition 12.12. Let V be a finite-dimensional real vector space.

(a) A covariant k-tensor $\alpha \in T^k(V^*)$ on V is said to be symmetric if its value is unchanged by interchanging any pair of its arguments; namely, for all $v_1, \dots, v_k \in V$ and all $1 \le i < j \le k$, we have

$$\alpha(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = \alpha(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k).$$

(b) The set of symmetric covariant k-tensors on V is denoted by $\Sigma^k(V^*)$. It is clearly a linear subspace of $T^k(V^*)$. By convention, we define $\Sigma^0(V^*) := \mathbb{R}$, and we also note that $\Sigma^1(V^*) = T^1(V^*) = V^*$.

Exercise 12.13. We define a projection $\mathrm{Sym}\colon T^k(V^*) \to \Sigma^k(V^*)$, called symmetrization, by the formula

$$\mathrm{Sym}(\alpha) := \frac{1}{k!} \sum_{\sigma \in S_k} {}^{\sigma}\alpha,$$

where ${}^{\sigma}\alpha$ was defined in Definition 12.11. Show that Sym is well-defined and linear, and that the following are equivalent:

(a) α is symmetric,

(b) $\alpha = {}^{\sigma}\alpha$ for all $\sigma \in S_k$,

(c) $\alpha = \mathrm{Sym}(\alpha)$.

12.4 Alternating Tensors

Recall that the determinant may be regarded as a function $\det\colon \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$, taking as input n column vectors with n entries each, and having as output the determinant of the $n \times n$ matrix formed by these n column vectors. This map is multilinear, so det is a covariant n-tensor on $\mathbb{R}^n$. Moreover, it has the property that its value changes sign whenever two of its input entries are interchanged; in other words, det is an alternating n-tensor. We now generalize this notion to arbitrary covariant k-tensors.

Definition 12.14. Let V be a finite-dimensional real vector space.

(a) A covariant k-tensor $\alpha \in T^k(V^*)$ on V is said to be alternating (or anti-symmetric or skew-symmetric) if its value changes sign whenever any two of its arguments are interchanged; namely, for all $v_1, \ldots, v_k \in V$ and $1 \leq i < j \leq k$, we have

$$\alpha(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\alpha(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k).$$

(b) The set of alternating covariant k-tensors on V is denoted by $\Lambda^k(V^*)$. It is clearly a linear subspace of $T^k(V^*)$ and its elements of $\Lambda^k(V^*)$ are also called exterior forms, multicovectors or k-covectors. By convention, we define $\Lambda^0(V^*) := \mathbb{R}$, and we also note that $\Lambda^1(V^*) = T^1(V^*) = V^*$.

Note that every covariant 2-tensor β can be expressed as a sum of an alternating and a symmetric tensor, because

$$\begin{aligned}\beta(v,w) &= \frac{1}{2}(\beta(v,w)-\beta(w,v)) + \frac{1}{2}(\beta(v,w)+\beta(w,v)) \\ &= \alpha(v,w)+\sigma(v,w),\end{aligned}$$

where

$$\alpha(v,w) \coloneqq \frac{1}{2}(\beta(v,w)-\beta(w,v)) \in \Lambda^2(V^*)$$

is an alternating 2-tensor on V and

$$\sigma(v,w) \coloneqq \frac{1}{2}(\beta(v,w)+\beta(w,v)) \in \Sigma^2(V^*)$$

is a symmetric 2-tensor on V. However, this is not true for tensors of higher rank, as the following exercise demonstrates.

Exercise 12.15. Let (e^1, e^2, e^3) be the standard dual basis for $(\mathbb{R}^3)^*$. Show that $e^1 \otimes e^2 \otimes e^3$ is not equal to a sum of an alternating tensor and a symmetric tensor.

Recall that there is a group homomorphism $\operatorname{sgn}\colon S_k \to \{\pm 1\}$, which maps a permutation $\sigma \in S_k$ to 1 if it is a product of an even number of transpositions (even permutation), and to -1 otherwise (odd permutation). We may use it to describe alternating tensors as follows.

Exercise 12.16. We define a projection $\operatorname{Alt}\colon T^k(V^*) \to \Lambda^k(V^*)$, called alternation, by the formula

$$\operatorname{Alt}(\alpha) \coloneqq \frac{1}{k!}\sum_{\sigma\in S_k}(\operatorname{sgn}\sigma)\,{}^{\sigma}\alpha,$$

where ${}^{\sigma}\alpha$ was defined in Definition 12.11. Show that Alt is well-defined and linear, and that the following are equivalent:

(a) α is alternating,

(b) $\alpha = (\operatorname{sgn}\sigma)\,{}^{\sigma}\alpha$ for all $\sigma \in S_k$,

(c) $\alpha = \operatorname{Alt}(\alpha)$,

(d) $\alpha(v_1,\dots,v_k) = 0$ whenever $v_1,\dots,v_k \in V$ are linearly dependent,

(e) $\alpha(v_1,\dots,v_k) = 0$ whenever there are $i \neq j$ such that $v_i = v_j$.

Example 12.17. Let us explicitly compute Alt for 1-, 2- and 3-tensors.

- If α is a 1-tensor, then $\operatorname{Alt}(\alpha) = \alpha$.
- If β is a 2-tensor, then
$$\operatorname{Alt}(\beta)(u,v) = \frac{1}{2}(\beta(u,v)-\beta(v,u)).$$

- If γ is a 3-tensor, then

$$\begin{aligned}\operatorname{Alt}(\gamma)(u,v,w) = \frac{1}{6}\big(&\gamma(u,v,w)+\gamma(v,w,u)+\gamma(w,u,v)\\ &-\gamma(v,u,w)-\gamma(u,w,v)-\gamma(w,v,u)\big).\end{aligned}$$

12.4.1 Elementary Alternating Tensors

Recall that for any basis of V, we described an induced basis of $T^k(V^*)$ in terms of tensor products of elements of the dual basis; cf. Proposition 12.8. We obtain here a similar description for a basis of $\Lambda^k(V^*)$.

Let V be a real vector space of dimension n, let $(E_1,\ldots,E_n)$ be a basis for V, and denote by $(\varepsilon^1,\ldots,\varepsilon^n)$ the corresponding dual basis for V^*. For a multi-index $I=(i_1,\ldots,i_k)\in[n]^{[k]}$, define the elementary alternating k-tensor (or elementary k-covector) ε^I by the formula

$$\varepsilon^I := k!\operatorname{Alt}\left(\varepsilon^{\otimes I}\right),$$

where

$$\varepsilon^{\otimes I}=\varepsilon^{i_1}\otimes\cdots\otimes\varepsilon^{i_k}\in T^k(V^*)$$

is the elementary k-tensor. Therefore, if $v_1,\ldots,v_k\in V$, then the value of ε^I at the k-tuple $(v_1,\ldots,v_k)$ is given by the formula

$$\begin{aligned}\varepsilon^I(v_1,\ldots,v_k) &= \sum_{\sigma\in S_k}(\operatorname{sgn}\sigma)\,\varepsilon^{\otimes I}\left(v_{\sigma(1)},\ldots,v_{\sigma(k)}\right)\\ &= \sum_{\sigma\in S_k}(\operatorname{sgn}\sigma)\prod_{1\le j\le k}\varepsilon^{i_j}\left(v_{\sigma(j)}\right)\\ &= \det\begin{pmatrix}\varepsilon^{i_1}(v_1) & \cdots & \varepsilon^{i_1}(v_k)\\ \vdots & \ddots & \vdots\\ \varepsilon^{i_k}(v_1) & \cdots & \varepsilon^{i_k}(v_k)\end{pmatrix}.\end{aligned}$$

In other words, to compute $\varepsilon^I(v_1,\ldots,v_k)$, we write the coefficients of $(v_1,\ldots,v_k)$ with respect to the basis $(E_1,\ldots,E_n)$ of V in the form of a $n\times k$-matrix, we consider the $k\times k$ submatrix formed by the lines $i_1,\ldots,i_k$, and then we compute its determinant.

Example 12.18. In terms of the standard dual basis (e^1,e^2,e^3) for $(\mathbb{R}^3)^*$, we have

$$e^{13}(v,w)=\det\begin{pmatrix}v^1 & w^1\\ v^3 & w^3\end{pmatrix}=v^1w^3-v^3w^1,$$

since $v=v^1e_1+v^2e_2+v^3e_3$ and $w=w^1e_1+w^2e_2+w^3e_3$, and

$$e^{123}(v,w,z)=\det(v,w,z).$$

Since $\mathrm{Alt}\colon T^k(V^*) \to \Lambda^k(V^*)$ is surjective, we know that $\left\{\varepsilon^I \mid I \in [n]^{[k]}\right\}$ is a generating set of $\Lambda^k(V^*)$. To extract from it a basis of $\Lambda^k(V^*)$, we need the following lemma, which describes the redundancy of $\left\{\varepsilon^I \mid I \in [n]^{[k]}\right\}$. In order to state it nicely, we need to introduce the following notation: for a multi-index $I \in [n]^{[k]}$ and a permutation $\sigma \in S_k$, denote by I_σ the multi-index

$$I_\sigma = \big(i_{\sigma(1)}, \ldots, i_{\sigma(k)}\big).$$

Also, denote by δ^I_J the following generalization of the Kronecker-delta to multi-indices $I, J \in [n]^{[k]}$:

$$\delta^I_J := \begin{cases} \operatorname{sgn}\sigma & \text{if neither } I \text{ nor } J \text{ have repeated entries and } J = I_\sigma \text{ for some } \sigma \in S_k, \\ 0 & \text{if } I \text{ or } J \text{ have repeated entries or } J \text{ is not a permutation of } I. \end{cases}$$

and observe that

$$\delta^I_J = \det\begin{pmatrix} \delta^{i_1}_{j_1} & \cdots & \delta^{i_1}_{j_k} \\ \vdots & \ddots & \vdots \\ \delta^{i_k}_{j_1} & \cdots & \delta^{i_k}_{j_k} \end{pmatrix}.$$

Lemma 12.19. With the same notation as in the preceding paragraph, the following statements hold:

(a) If I has a repeated index, then $\varepsilon^I = 0$.

(b) If $J = I_\sigma$ for some $\sigma \in S_k$, then $\varepsilon^J = (\operatorname{sgn}\sigma)\,\varepsilon^I$.

(c) For $I, J \in [n]^{[k]}$ we have

$$\varepsilon^I(E_{j_1}, \ldots, E_{j_k}) = \delta^I_J.$$

Proof. Exercise! □

Lemma 12.19 tells us that from the generating set $\left\{\varepsilon^I \mid I \in [n]^{[k]}\right\}$ of $\Lambda^k(V^*)$, we may discard all those ε^I's for which I has a repeated index, and for any I having no repeated index, we need only take one element from the set $\{\varepsilon^{I_\sigma} \mid \sigma \in S_k\}$ and discard the rest. A nice choice is thus the following: notice that for any multi-index I having no repeated indices, there exists a unique permutation $\sigma \in S_k$ such that I_σ is strictly increasing, i.e., $i_{\sigma(1)} < \cdots < i_{\sigma(k)}$. Therefore, according to Lemma 12.19, the set $\left\{\varepsilon^I \mid I \in [n]^{[k]} \text{ is strictly increasing}\right\}$ still generates $\Lambda^k(V^*)$, and there is no obvious redundancy in it. Essentially due to Lemma 12.19(c), this set is linearly independent, and thus we obtain the following result:

Proposition 12.20. With the same notation as above, the set

$$\left\{\varepsilon^I \mid I \in [n]^{[k]} \text{ is a strictly increasing multi-index}\right\}$$

is a basis for $\Lambda^k(V^*)$. In particular, we have

$$\dim_{\mathbb{R}} \Lambda^k(V^*) = \binom{n}{k} = \frac{n!}{k!(n-k)!},$$

and

$$\Lambda^k(V^*) = \{0\} \quad \text{for } k > n.$$

Proof. Assume first that $k > n$. Since then every k-tuple of vectors is linearly dependent, it follows from Exercise 12.16(d) that $\Lambda^k(V^*) = \{0\}$.

Assume now that $k \leq n$. We need to show that

$$\mathcal{E} := \left\{ \varepsilon^I \mid I \in [n]^{[k]} \text{ is a strictly increasing multi-index} \right\}$$

is linearly independent and spans $\Lambda^k(V^*)$. The fact that $\mathcal{E}$ generates $\Lambda^k(V^*)$ was already discussed above. Suppose now that we have some linear relation

$$\sum_{I \in [n]^{[k]} \text{ strictly increasing}} \lambda_I \, \varepsilon^I = 0$$

for some $\lambda_I \in \mathbb{R}$. If we fix a strictly increasing multi-index $J \in [n]^{[k]}$, then evaluating the above relation at $(E_{j_1}, \ldots, E_{j_k})$ gives $\lambda_J = 0$ according to Lemma 12.19(c). Thus, $\mathcal{E}$ is linearly independent. In conclusion, $\mathcal{E}$ is a basis of $\Lambda^k(V^*)$, as desired. □

In particular, if V is a real vector space of dimension n, then the above proposition implies that $\Lambda^n(V^*)$ is 1-dimensional, spanned by the elementary n-covector $\varepsilon^{(1,\ldots,n)}$. As discussed in the beginning of this subsection, $\varepsilon^{(1,\ldots,n)}$ sends an n-tuple $(v_1, \ldots, v_n)$ to the determinant of the matrix $(v^i_j)_{1 \leq i,j, \leq n}$, where $v^i_j = \varepsilon^i(v_j)$ is the i-th component of v_j with respect to the chosen basis of V. Note that when $V = \mathbb{R}^n$ with the standard basis, the covector $\varepsilon^{(1,\ldots,n)}$ (which by definition is a function from $(\mathbb{R}^n)^n = \mathbb{R}^{n^2}$ to $\mathbb{R}$) is precisely the usual determinant function.

One consequence of this observation is the following useful description of the behavior of an n-covector on an n-dimensional vector space under linear maps. Recall that if $T \colon V \to V$ is a linear map, then the determinant of T is defined to be the determinant of the matrix representation of T with respect to any basis (recall that any two such matrix representation are conjugations of each other and hence have the same determinant, so this is well-defined).

Proposition 12.21. Let V be an n-dimensional real vector space and let $\omega \in \Lambda^n(V^*)$. If $T \colon V \to V$ is any linear map and if $v_1, \ldots, v_n \in V$ are arbitrary vectors, then

$$\omega(Tv_1, \ldots, Tv_n) = (\det T)\, \omega(v_1, \ldots, v_n). \tag{$\bullet$}$$

Proof. Let (E_i) be any basis for V, and let (ε^i) be the dual basis. Denote by $(T^j_i)_{1 \leq i,j \leq n}$ the matrix of T with respect to this basis, and set $T_i = TEi = \sum_j T^j_i E_j$. By Proposition 12.20, we can write $\omega = c\varepsilon^{(1,\ldots,n)}$ for some $c \in \mathbb{R}$. Since both sides of $(\bullet)$ are multilinear functions of $(v_1, \ldots, v_n)$, it suffices to verify the identity when the v_i's are basis vectors. Furthermore, since both sides are alternating, by Lemma 12.19 we only need to check the case $(v_1, \ldots, v_n) = (E_1, \ldots, E_n)$. In this case, the right-hand side of $(\bullet)$ is

$$(\det T)\, c\, \varepsilon^{(1,\ldots,n)}(E_1, \ldots, E_n) = c \,\det T.$$

On the other hand, the left-hand side of $(\bullet)$ reduces to

$$\omega(TE_1, \ldots, TE_n) = c\, \varepsilon^{(1,\ldots,n)}(T_1, \ldots, T_n) = c\, \det\left((\varepsilon^j(T_i))_{1\le i,j\le n}\right) = c\, \det\left((T_i^j)_{1\le i,j\le n}\right).$$

which is thus equal to the right-hand side. □

12.4.2 The Wedge Product

Recall that for any covariant tensors $\alpha \in T^k(V^*)$ and $\beta \in T^l(V^*)$ we defined the covariant $(k+l)$-tensor $\alpha \otimes \beta$; see Definition 12.6. This allowed us to build 'higher' covariant tensors out of lower ones, and also to describe a basis for $T^k(V^*)$ in terms of tensor products of elements of a dual basis. We now describe a similar construction for alternating tensors.

Definition 12.22. Let V be a finite-dimensional real vector space, and let $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$ be alternating tensors on V. The wedge product (or exterior product) of ω and η is denoted by $\omega \wedge \eta$ and is defined to be the $(k+l)$-covector given by the formula

$$\omega \wedge \eta := \frac{(k+l)!}{k!l!}\,\mathrm{Alt}(\omega \otimes \eta).$$

As $\otimes$ is bilinear and Alt is linear, the map $-\wedge-\colon \Lambda^k(V^*) \times \Lambda^l(V^*) \to \Lambda^{k+l}(V^*)$ is bilinear. It is therefore natural to examine what the wedge product looks like on basis vectors. This also motivates the somewhat mysterious normalization factor $(k+l)!/(k!l!)$, because we have the following result.

Lemma 12.23. Let V be a finite-dimensional real vector space, and let $(\varepsilon^1, \ldots, \varepsilon^n)$ be a basis for V^*. For any multi-indices $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_l)$ we have the formula

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{I \frown J},$$

where $I \frown J = (i_1, \ldots, i_k, j_1, \ldots, j_l)$ is the $(k+l)$-multi-index obtained by concatenating I and J.

Proof. By multilinearity, as in the proof of Proposition 12.8, it suffices to show that

$$\varepsilon^I \wedge \varepsilon^J(E_{p_1}, \ldots, E_{p_{k+l}}) = \varepsilon^{I\frown J}(E_{p_1}, \ldots, E_{p_{k+l}}) \tag{$\star$}$$

for any sequence of basis vectors $(E_{p_1}, \ldots, E_{p_{k+l}})$. We do this by considering several cases.

Case 1: The multi-index $P = (p_1, \ldots, p_{k+l})$ has a repeated index. Then by part (e) of Exercise 12.16, both sides of $(\star)$ evaluate to 0.

Case 2: <u>P contains an index that does not appear in either I or J.</u> In this case, the right-hand side of $(\star)$ is zero by part (c) of Lemma 12.19. Similarly, each term in the expansion of the left-hand side of $(\star)$ involves either I or J evaluated on a sequence of basis vectors that is not a permutation of I or J, respectively, so the left-hand side is also zero.

Case 3: <u>$P = I \frown J$ and P has no repeated indices.</u> In this case, the right-hand side of $(\star)$ is equal to 1, again by part (c) of Lemma 12.19, so we need to show that the left-hand side is also equal to 1. By definition,

$$\begin{aligned}
&\varepsilon^I \wedge \varepsilon^J(E_{p_1}, \dots, E_{p_{k+l}}) = \\
&\quad = \frac{(k+l)!}{k!l!} \operatorname{Alt}(\varepsilon^I \otimes \varepsilon^J) \\
&\quad = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) \varepsilon^I(E_{p_{\sigma(1)}}, \dots, E_{p_{\sigma(k)}}) \varepsilon^J(E_{p_{\sigma(k+1)}}, \dots, E_{p_{\sigma(k+l)}}).
\end{aligned}$$

By Lemma 12.19 again, the only terms in the sum above that give nonzero values are those in which σ permutes the first k indices and the last l indices of P separately. In other words, σ must be of the form $\sigma = \tau\eta$, where $\tau \in S_k$ acts by permuting $\{1, \dots, k\}$ and $\eta \in S_l$ acts by permuting $\{k+1, \dots, k+l\}$. Since then $\operatorname{sgn} \sigma = (\operatorname{sgn} \tau)(\operatorname{sgn} \eta)$, we have

$$\begin{aligned}
&\varepsilon^I \wedge \varepsilon^J(E_{p_1}, \dots, E_{p_{k+l}}) = \\
&= \frac{1}{k!l!} \sum_{\substack{\tau \in S_k \\ \eta \in S_l}} (\operatorname{sgn} \tau)(\operatorname{sgn} \eta)\, \varepsilon^I(E_{p_{\tau(1)}}, \dots, E_{p_{\tau(k)}})\, \varepsilon^J(E_{p_{k+\eta(1)}}, \dots, E_{p_{k+\eta(l)}}) \\
&= \left(\frac{1}{k!} \sum_{\tau \in S_k} (\operatorname{sgn} \tau)\, \varepsilon^I(E_{p_{\tau(1)}}, \dots, E_{p_{\tau(k)}}) \right) \left(\frac{1}{l!} \sum_{\eta \in S_l} (\operatorname{sgn} \eta)\, \varepsilon^J(E_{p_{k+\eta(1)}}, \dots, E_{p_{k+\eta(l)}}) \right) \\
&= \Big(\operatorname{Alt}(\varepsilon^I)(E_{p_1}, \dots, E_{p_k}) \Big) \Big(\operatorname{Alt}(\varepsilon^J)(E_{p_{k+1}}, \dots, E_{p_{k+l}}) \Big) \\
&= \varepsilon^I(E_{p_1}, \dots, E_{p_k})\, \varepsilon^J(E_{p_{k+1}}, \dots, E_{p_{k+l}}) \\
&= 1
\end{aligned}$$

where we used that Alt fixes alternating tensors by Exercise 12.16, and again used part (c) of Lemma 12.19 (recall that we are in the case $P = I \frown J$).

Case 4: <u>P is a permutation of $I \frown J$ and has no repeated indices.</u> In this case, applying a permutation to P brings us back to Case 3. As both sides of $(\star)$ are alternating, the effect of this permutation is to multiply both sides by the same sign. Hence the result holds in this final case as well.

This completes the proof of the lemma. $\square$

Together with the bilinearity of $- \wedge -$, this gives the following properties of the wedge product.

Proposition 12.24. Let ω, η, ξ be multicovectors on a finite-dimensional real vector space V. Then we have the following properties:

(a) Associativity:
$$\omega \wedge (\eta \wedge \xi) = (\omega \wedge \eta) \wedge \xi.$$

(b) Anticommutativity: if $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$, then
$$\omega \wedge \eta = (-1)^{kl}\, \eta \wedge \omega.$$

(c) If $(\varepsilon^1, \ldots, \varepsilon^n)$ is a basis of V^* and $I = (i_1, \ldots, i_k)$ a multi-index, then
$$\varepsilon^{i_1} \wedge \ldots \wedge \varepsilon^{i_k} = \varepsilon^I.$$

(d) For any $\omega^1, \ldots, \omega^k \in V^*$ and $v_1, \ldots, v_k \in V$ we have
$$\omega^1 \wedge \ldots \wedge \omega^k(v_1, \ldots, v_k) = \det\left(\left(\omega^j(v_i)\right)_{1 \le i,j \le k}\right).$$

Proof. Exercise! □

Due to Proposition 12.24(c), we generally use the notations ε^I and $\varepsilon^{i_1} \wedge \ldots \wedge \varepsilon^{i_k}$ interchangeably.

An element $\eta \in \Lambda^k(V^*)$ is said to be decomposable if it can be expressed in the form $\eta = \omega^1 \wedge \ldots \wedge \omega^k$ for some covectors $\omega^1, \ldots, \omega^k \in V^*$. Note that not every k-covector is decomposable when $k > 1$; however, it follows from Proposition 12.20 and Proposition 12.24(c) that every k-covector can be written as a linear combination of decomposable ones.

Chapter 13

Orientations

13.1 Orientations of Vector Spaces

We begin with orientations of vector spaces. We are all familiar with certain informal rules for singling out preferred ordered bases of $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$. We usually choose a basis for $\mathbb{R}^1$ that points to the right (i.e., in the positive direction). A natural family of preferred ordered bases for $\mathbb{R}^2$ consists of those for which the rotation from the first vector to the second is in the counterclockwise direction. And every student of vector calculus encounters "right-handed" bases in $\mathbb{R}^3$: these are the ordered bases (E_1, E_2, E_3) with the property that when the fingers of your right hand curl from E_1 to E_2, your thumb points in the direction of E_3.

Although "to the right", "counterclockwise", and "right-handed" are not mathematical terms, it is easy to translate the rules for selecting preferred bases of $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$ into rigorous mathematical language: you can check that in all three cases, the preferred bases are the ones whose transition matrices from the standard basis have positive determinants.

In an abstract vector space for which there is no canonical basis, we no longer have any way to determine which bases are "correctly oriented". For example, if V is the vector space of polynomials in one real variable of degree at most 2, who is to say which of the ordered bases $(1, x, x^2)$ and $(x^2, x, 1)$ is "right-handed"? All we can say in general is what it means for two bases to have the "same orientation". Thus we are led to introduce the following definition.

Definition 13.1. Let V be a real vector space of dimension $n \geq 1$. We say that two ordered bases $(E_1, \ldots, E_n)$ and $(\widetilde{E}_1, \ldots, \widetilde{E}_n)$ for V are consistently oriented if the transition matrix $(B_i^j)_{1 \leq i,j \leq n}$, defined by

$$E_i = \sum_j B_i^j \widetilde{E}_j,$$

has positive determinant.

Exercise 13.2. Show that being consistently oriented is an equivalence relation on the set of all ordered bases of V, and show that there are exactly two equivalence classes.

Definition 13.3. Let V be a real vector space. If $\dim_{\mathbb{R}} V = n \geq 1$, we define an orientation for V

as an equivalence class of ordered bases. If $(E_1, \dots, E_n)$ is any ordered basis for V, we denote the orientation that it determines by $[E_1, \dots, E_n]$, and the opposite orientation by $-[E_1, \dots, E_n]$. On the other hand, for the special case of a zero-dimensional vector space V, we define an orientation of V to be simply a choice of one of the numbers ± 1.

Definition 13.4. A vector space together with a choice of orientation is called an oriented vector space. If V is oriented, then any ordered basis $(E_1, \dots, E_n)$ that is in the given orientation is said to be positively oriented (or simply oriented). Any ordered basis that is not in the given orientation is said to be negatively oriented.

Example 13.5. Consider the Euclidean space $V = \mathbb{R}^n$. The orientation $[e_1, \dots, e_n]$ of $\mathbb{R}^n$ determined by the standard basis $\{e_1, \dots, e_n\}$ is called the standard orientation. You should convince yourself that, in our usual way of representing the axes graphically, an oriented basis for $\mathbb{R}^1$ is one that points to the right; an oriented basis for $\mathbb{R}^2$ is one for which the rotation from the first basis vector to the second is counterclockwise; and an oriented basis for $\mathbb{R}^3$ is a right-handed one. (These can be taken as mathematical definitions for the words "right", "counterclockwise", and "right-handed".) The standard orientation for $\mathbb{R}^0$ is defined to be $+1$.

There is an important connection between orientations and alternating tensors, which is expressed in the following proposition.

Proposition 13.6. Let V be a real vector space of dimension n. Each nonzero element $\omega \in \Lambda^n(V^*)$ determines an orientation $\mathcal{O}_\omega$ of V as follows: if $n \geq 1$, then $\mathcal{O}_\omega$ is the set of ordered bases $(E_1, \dots, E_n)$ for V such that $\omega(E_1, \dots, E_n) > 0$, while if $n = 0$, then $\mathcal{O}_\omega$ is $+1$ if $\omega > 0$, and -1 if $\omega < 0$. Moreover, two nonzero n-covectors on V determine the same orientation if and only if each is a positive multiple of the other.

Proof. The 0-dimensional case is immediate, since a nonzero element of $\Lambda^0(V^*)$ is just a nonzero real number (as it is a function $\mathbb{R}^0 \to \mathbb{R}$). Thus, we may assume that $n \geq 1$. Let ω be a nonzero element of $\Lambda^n(V^*)$, and denote by $\mathcal{O}_\omega$ the set of ordered bases on which ω gives positive values. We need to show that $\mathcal{O}_\omega$ is exactly one equivalence class. Suppose (E_i) and $(\widetilde{E}_j)$ are any two ordered bases for V, and let $B\colon V \to V$ be the linear map sending E_j to $\widetilde{E}_j$ for all j. This means that the matrix of B with respect to (E_i) on the source and $(\widetilde{E}_j)$ on the target is the transition matrix between the two bases. By the [PDF: Multilinear Algebra, Proposition 21], we obtain

$$\omega(\widetilde{E}_1, \dots, \widetilde{E}_n) = \omega(BE_1, \dots, BE_n) = (\det B)\, \omega(E_1, \dots, E_n).$$

It follows that the basis $(\widetilde{E}_j)$ is consistently oriented with (E_i) if and only if $\omega(\widetilde{E}_1, \dots, \widetilde{E}_n)$ and $\omega(E_1, \dots, E_n)$ have the same sign, which is the same as saying that $\mathcal{O}_\omega$ is one equivalence class. The last statement then follows easily (and is thus left as an exercise). □

Definition 13.7. If V is an oriented n-dimensional real vector space and if ω is an n-covector that determines the orientation of V as described in the above proposition, then we say that ω is a (positively) oriented n-covector.

For example, the n-covector $\varepsilon^{1\dots n} = \varepsilon^1 \wedge \cdots \wedge \varepsilon^n$ is positively oriented for the standard orientation on $\mathbb{R}^n$.

Recall that if V is an n-dimensional real vector space, then the space $\Lambda^n(V^*)$ is 1-dimensional. Proposition 13.6 shows that choosing an orientation for V is equivalent to choosing one of the two components of $\Lambda^n(V^*) \setminus \{0\}$. This formulation also works for 0-dimensional vector spaces, and explains why we have defined an orientation of a 0-dimensional space in the way we did.

13.2 Orientations of Smooth Manifolds

In this section we briefly discuss the theory of orientations of smooth manifolds. They have numerous applications, most notably in the theory of integration on manifolds, which will be presented in Lecture 14 of this course.

Definition 13.8. Let M be a smooth manifold with or without boundary. A pointwise orientation on M is defined to be a choice of orientation of each tangent space.

By itself, this is not a very useful concept, because the orientations at nearby points may have no relation to each other. For example, a pointwise orientation on $\mathbb{R}^n$ might switch randomly from point to point between the standard orientation and its opposite. In order for pointwise orientations to have some relationship with the smooth structure, we need an extra condition to ensure that the orientations of nearby tangent spaces are consistent with each other.

Definition 13.9. Let M be a smooth manifold with or without boundary, endowed with a pointwise orientation. If (E_i) is a local frame for TM over an open subset $U \subseteq M$, then we say that (E_i) is positively oriented (or simply oriented) if $(E_1|_p, \ldots, E_n|_p)$ is a positively oriented ordered basis for T_pM at each point $p \in U$; see Definition 13.4. A negatively oriented frame for TM over $U \subseteq M$ is defined analogously.

Definition 13.10. Let M be a smooth manifold with or without boundary.

(a) A pointwise orientation on M is said to be continuous if every point of M is in the domain of an oriented local frame for TM.

(a) An orientation of M is a continuous pointwise orientation.

(a) We say that M is orientable if there exists an orientation for it; otherwise we say that M is nonorientable.

Example 13.11. We give here some examples of orientable and nonorientable manifolds.

(a) Every parallelizable[1] manifold is orientable. Indeed, if $(E_1, \ldots, E_n)$ is a smooth global frame for M, then we define a pointwise orientation on M by declaring the basis $(E_1|_p, \ldots, E_n|_p)$ for T_pM to be positively oriented at each $p \in M$, and it is clear that this pointwise orientation is continuous, because every point of M is in the domain of the oriented smooth global frame (E_i). Therefore, for each $n \in \mathbb{N}$, the Euclidean space $\mathbb{R}^n$ is orientable.

(a) For each $n \in \mathbb{N}$, the unit n-sphere $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$ is orientable. Indeed, this follows from Proposition 13.20 (because $\mathbb{S}^n$ is a hypersurface in $\mathbb{R}^{n+1}$, to which the vector field $N = x^i \partial/\partial x^i$ is nowhere tangent) or Proposition 13.22 (because $\mathbb{S}^n$ is the boundary of the closed unit ball).

(a) The so-called Möbius band is nonorientable.

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[1] A smooth manifold M with or without boundary which admits a smooth global frame or, equivalently, whose tangent bundle TM is the trivial smooth vector bundle of rank $\dim M$ (see *Exercise* 5*, Sheet* 10) is called parallelizable. Note that the Euclidean space $\mathbb{R}^n$ is parallelizable, and it can also be shown that $\mathbb{S}^1$, $\mathbb{S}^3$ and $\mathbb{S}^7$ are the only spheres that are parallelizable.

Definition 13.12. An oriented manifold (with or without boundary) is an ordered pair $(M, \mathcal{O})$, where M is an orientable smooth manifold (with or without boundary) and $\mathcal{O}$ is a choice of orientation for M. For each $p \in M$, the orientation of T_pM determined by $\mathcal{O}$ is denoted by $\mathcal{O}_p$.

If M is zero-dimensional, then this definition just means that an orientation of M is a choice of ± 1 attached to each of its points. The continuity condition is vacuous in this case, and the notion of oriented frames is not useful. Clearly, every 0-manifold is orientable.

Exercise 13.13. Let M be an oriented smooth manifold with or without boundary of dimension $n \geq 1$. Show that every local frame with connected domain is either positively oriented or negatively oriented. Moreover, show that the connectedness assumption is necessary.

13.2.1 Two Ways of Specifying Orientations

The following two propositions, namely Proposition 13.14 and Proposition 13.18, give ways of specifying orientations on manifolds that are more practical to use than the definition.

Proposition 13.14 (The orientation determined by an n-form)**.** Let M be a smooth n-manifold with or without boundary. Any non-vanishing n-form ω on M determines a unique orientation of M for which ω is positively oriented at each point. Conversely, if M is given an orientation, then there is a smooth non-vanishing n-form on M that is positively oriented at each point.

Proof. Let ω be a non-vanishing n-form on M. By Proposition 13.6, ω defines a pointwise orientation on M, so it remains to show that it is continuous. Since this is trivially true for $n = 0$, we may assume that $n \geq 1$. Given $p \in M$, let (E_i) be any local frame for TM over a connected open neighborhood U of p in M, and let (ε^i) be the dual coframe. On U, the expression for ω in this frame is

$$\omega = f\, \varepsilon^1 \wedge \ldots \wedge \varepsilon^n$$

for some continuous function f on U. The fact that ω is non-vanishing means that f is non-vanishing, and thus

$$\omega_p\left(E_1|_p, \ldots, E_n|_p\right) = f(p) \neq 0 \quad \text{for all } p \in U.$$

Since U is connected, it follows that this expression is either always positive or always negative on U, and therefore the given frame is either positively oriented or negatively oriented. If the latter case holds, then we can replace E_1 by $-E_1$ to obtain a new frame that is positively oriented. Hence, the pointwise orientation determined by ω is continuous.

The proof of the converse uses partitions of unity and is thus omitted. □

Due to Proposition 13.14, we may now give the following definition.

Definition 13.15. Let M be a smooth n-manifold with or without boundary. Any non-vanishing n-form on M is called an orientation form. If M is oriented and if ω is an orientation form determining the given orientation, then we also say that ω is positively oriented (or simply oriented).

If M is zero-dimensional, then a non-vanishing 0-form (i.e., a non-vanishing smooth real-valued function) on M assigns the orientation $+1$ to points where it is positive and -1 to points where it is negative.

Remark 13.16. It is easy to check that if ω and $\widetilde{\omega}$ are two positively oriented smooth n-forms on M, then $\widetilde{\omega} = f\omega$ for some strictly positive smooth real-valued function f on M; see Proposition 13.6.

Definition 13.17. (a) A smooth atlas $\{(U_\alpha, \varphi_\alpha)\}$ for a smooth manifold M with or without boundary is said to be consistently oriented if for each α, β, the transition map $\varphi_\beta \circ \varphi_\alpha^{-1}$ has positive Jacobian determinant everywhere on $\varphi_\alpha(U_\alpha \cap U_\beta)$.

(a) A smooth coordinate chart $(U, (x^i))$ on an oriented smooth manifold with or without boundary is said to be positively oriented (or simply oriented) if the coordinate frame $(\partial/\partial x^i)$ is positively oriented, and negatively oriented if the coordinate frame $(\partial/\partial x^i)$ is negatively oriented.

Proposition 13.18 (The orientation determined by a coordinate atlas)**.** Let M be a smooth manifold with or without boundary of dimension $n \geq 1$. Given any consistently oriented smooth atlas for M, there exists a unique orientation for M with the property that each chart in the given atlas is positively oriented. Conversely, if M is oriented and either $\partial M = \emptyset$ or $n > 1$, then the collection of all oriented smooth charts is a consistently oriented atlas for M.

Proof. Assume first that M has a consistently oriented smooth atlas. Each chart in the atlas determines a pointwise orientation at each point of its domain. Wherever two of the charts overlap, the transition matrix between their respective coordinate frames is the Jacobian matrix of the transition map (see the solution to part (c) of *Exercise* 1*, Sheet* 10), which has positive determinant by assumption, so they determine the same pointwise orientation at each point. The pointwise orientation on M thus determined is continuous, because each point of M is in the domain of an oriented coordinate frame.

Conversely, assume that M is oriented and either $\partial M = \emptyset$ or $n > 1$. Each point is in the domain of a smooth chart with connected domain, and if the chart is negatively oriented (see Exercise 13.13), then we can replace x^1 with $-x^1$ to obtain a new chart that is positively oriented. The fact that all these charts are positively oriented guarantees that their transition maps have positive Jacobian determinants, so they form a consistently oriented atlas.[2] □

Exercise 13.19. Let M be a connected, orientable, smooth manifold with or without boundary. Show that M has exactly two orientations. Moreover, if two orientations of M agree at one point, then they are equal.

13.2.2 Orientations of Hypersurfaces

If M is an oriented smooth manifold and if S is a smooth immersed submanifold of M (with or without boundary), then S might not inherit an orientation from M, even if S is embedded. Clearly,

[2]This does not work for boundary charts when $\dim M = n = 1$, because of our convention that the last coordinate is nonnegative in a boundary chart.

it is not sufficient to restrict an orientation form from M to S, since the restriction of an n-form to a manifold of lower dimension must necessarily be zero. For example, the so-called Möbius band is nonorientable, even though it can be embedded in $\mathbb{R}^3$, which is orientable.

However, when S is an immersed or embedded hypersurface in M (i.e., a codimension 1-submanifold of M), it is sometimes possible to use an orientation on M to induce an orientation on S; see Proposition 13.20 below for the details. Recall first that a vector field along S is a section of the ambient tangent bundle $TM|_S$, i.e., a continuous map $N\colon S \to TM$ with the property that $N_p \in T_pM$ for every $p \in S$, and that such a vector field is said to be nowhere tangent to S if $N_p \in T_pM \setminus T_pS$ for all $p \in S$; cf. Proposition 7.8. Note that any vector field on M restricts to a vector field along S, but in general, not every vector field along S is of this form.

Proposition 13.20. Let M be an oriented smooth n-manifold with or without boundary, let S be an immersed hypersurface with or without boundary in M, and let N be a vector field along S which is nowhere tangent to S. Then S has a unique orientation such that for each $p \in S$, $(E_1, \ldots, E_{n-1})$ is an oriented basis for T_pS if and only if $(N_p, E_1, \ldots, E_{n-1})$ is an oriented basis for T_pM.

Note that not every hypersurface admits a nowhere tangent vector field. However, the following result gives a sufficient condition that holds in many cases.

Corollary 13.21. If M is an oriented smooth manifold and if $S \subseteq M$ is a regular level set of a smooth function $f\colon M \to \mathbb{R}$, then S is orientable.

13.2.3 Boundary Orientations

If M is a smooth manifold with boundary, then its boundary ∂M is an embedded hypersurface (without boundary) in M (see the Theorem in Remark 9.7(4)) and there always exists a smooth outward-pointing vector field along ∂M (see the second Proposition in Remark 9.7(5)). Since such a vector field is nowhere tangent to ∂M (see the first Proposition in Remark 9.7(5)), it determines an orientation on ∂M by Proposition 13.20, provided that M is oriented. The following proposition shows that this orientation is independent of the choice of an outward-pointing vector field along ∂M, and is called the induced orientation or the Stokes orientation on ∂M.

Proposition 13.22 (The induced orientation on a boundary)**.** Let M be an oriented smooth n-manifold with boundary, where $n \geq 1$. Then ∂M is orientable, and all outward-pointing vector fields along ∂M determine the same orientation on ∂M.

Example 13.23. We determine the induced orientation on $\partial\mathbb{H}^n$ when $\mathbb{H}^n$ itself has the standard orientation inherited from $\mathbb{R}^n$. We can identify $\partial\mathbb{H}^n$ with $\mathbb{R}^{n-1}$ under the correspondence

$$(x^1, \ldots, x^{n-1}, 0) \leftrightarrow (x^1, \ldots, x^{n-1}).$$

Since the vector field $-\partial/\partial x^n$ is outward-pointing along $\mathbb{H}^n$, the standard coordinate frame for $\mathbb{R}^{n-1}$ is positively oriented for $\partial\mathbb{H}^n$ if and only if $[-\partial/\partial x^n, \partial/\partial x^1, \ldots, \partial/\partial x^{n-1}]$ is the standard orientation for $\mathbb{R}^n$; see Proposition 13.20. This orientation satisfies

$$\begin{aligned}\left[-\partial/\partial x^n, \partial/\partial x^1, \ldots, \partial/\partial x^{n-1}\right] &= -\left[\partial/\partial x^n, \partial/\partial x^1, \ldots, \partial/\partial x^{n-1}\right] \\ &= (-1)^n \left[\partial/\partial x^1, \ldots, \partial/\partial x^{n-1}, \partial/\partial x^n\right].\end{aligned}$$

Thus, the induced orientation on $\partial\mathbb{H}^n$ is equal to the standard orientation on $\mathbb{R}^{n-1}$ when n is even, but it is opposite to the standard orientation when n is odd. In particular, the standard coordinates on $\partial\mathbb{H}^n \approx \mathbb{R}^{n-1}$ are positively oriented if and only if n is even.

13.2.4 Orientations and Smooth Maps

Definition 13.24. Let M and N be oriented smooth manifolds with or without boundary and let $F\colon M \to N$ be a local diffeomorphism. If both M and N are positive-dimensional, then we say that F is orientation-preserving if for each $p \in M$, the isomorphism dF_p takes positively oriented bases of T_pM to positively oriented bases of $T_{F(p)}N$, and orientation-reversing if it takes positively oriented bases of T_pM to negatively oriented bases of $T_{F(p)}N$. If, on the other hand, both M and N are zero-dimensional, then we say that F is orientation preserving if for every $p \in M$, the points p and $F(p)$ have the same orientation, and it is orientation reversing if they have opposite orientation; see the paragraph after Definition 13.12.

Note that a composition of orientation-preserving maps is also orientation-preserving.

Exercise 13.25. Let M and N be oriented positive-dimensional smooth manifolds with or without boundary and let $F\colon M \to N$ be a local diffeomorphism. Show that the following are equivalent:

(a) F is orientation-preserving.

(a) With respect to any oriented smooth charts for M and N, the Jacobian matrix of F has positive determinant.

(a) If ω is any positively oriented orientation form for N, then $F^*\omega$ is a positively oriented orientation form for M.

Proposition 13.26 (The pullback orientation)**.** Let M and N be smooth manifolds with or without boundary. If $F\colon M \to N$ is a local diffeomorphism and if N is oriented, then M has a unique orientation, called the pullback orientation induced by F, such that F is orientation-preserving.

Proof. For each $p \in M$ there is a unique orientation on T_pM that makes the isomorphism $dF_p\colon T_pM \to T_{F(p)}N$ orientation-preserving. This defines a pointwise orientation on M; provided that it is continuous, it is the unique orientation on M with respect to which F is orientation-preserving. To see that it is continuous, just choose a smooth orientation form ω of N using Proposition 13.14 (so that ω is positively oriented) and note that $F^*\omega$ is a smooth orientation form for M, determining by construction and by Proposition 13.14 the above pointwise orientation on M, which is thus continuous, as desired. □

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