Exam 16.June 2009
Neural Networks and Biological Modeling

- Write your name in readable letters on top of this page
- The exam lasts 160 minutes
- All responses must be on these exam sheets
- Except for one paper A4 of handwritten notes, no documentation is allowed.
- You may use a pocket calculator, but not a programmable computer
- The total number of points that can be achieved is 44

Evaluation
Section 1: ....../7 pts

Section 2: ....../12 pts

Section 3: ....../9 pts

Section 4: ....../10 pts

Section 5: ....../6 pts

...................../44
The exam has 10 pages, the back of the pages is also used!
QUESTION 1: ION CHANNEL

We consider the following model of a ion channel

\[ I_{ion} = g_0 x^p (u - E) \]

where \( u \) is the membrane potential. The parameters \( g_0, p \) and \( E = 0 \) are constants.

(a) What is the name of the variable \( E \)? .........................

Why does it have this name, what does it signify (give answer in one sentence)

..........................................................  /1 point

(b) The variable \( x \) follows the dynamics

\[ \frac{dx}{dt} = -\frac{x - x_0(u)}{\tau} \]

Suppose we make a voltage step from a fixed value \( E \) to a new constant value \( u_0 \). Give the mathematical solution \( x(t) \) for \( t > 0 \)

\[ x(t) = .......................\]  /2 points
An electrophysiologist tells you that he is able to apply voltage steps as in (b) and that by measuring the current he wants to determine the parameters $g_0$ and $x_0(u)$ of the ion channel in (a) and (b).

(c) How should he proceed to measure the parameter $p$? What would be different between the case $p=1$ and $p = 4$? You can sketch a little figure to illustrate your answer.

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(d) Under the assumption that $x_0(u)$ is bounded between zero and 1, how can he measure $g_0$?
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................................................................. /1 point

(e) Given the value of $g_0$ and $p$, how can he measure the value $x_0(u_1)$ at some arbitrary value $u_1$? If possible, give a mathematical expression to illustrate your explanation.
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3
**Question 2: Integrate-and-Fire Model/Phase Plane Analysis**

(12 points)

An integrate-and-fire neuron model with adaptation is described by the two differential equations

\[
\frac{du}{dt} = F(u) - w + I \\
\tau \frac{dw}{dt} = -w + a(u + 10)
\]

(1)

(2)

If \( u \geq 10 \) the variable \( u \) is reset to \( u = 0 \). The variable \( w \) is increased by an amount of 4 during reset.

We take

\[
F(u) = -(u + 10) \quad \text{for} \quad u \leq 0
\]

\[
F(u) = -10 + 5u \quad \text{for} \quad u > 0
\]

(3)

(4)

(a) Plot the nullclines in the phase plane \((u, w)\) for \( I = 0 \) and \( a = 0.5 \) using the space here:

/2 points

(b) In the same graph, add representative arrows indicating qualitatively the flow on the nullclines and in different regions of the phase plane (you may assume \( \tau = 2 \)). /2 points

(c) In the same graph, mark the rest state. /1 point

(d) In the same graph, indicate a trajectory in the phase plane, after a stimulus \( I(t) = 11\delta(t) \) has been applied starting from rest

[\( \delta \) denotes the Dirac delta function]. /1 point

(e) In the same graph, indicate a trajectory in the phase plane, after a stimulus \( I(t) = 15\delta(t) \) has been applied starting from rest

[\( \delta \) denotes the Dirac delta function]. /1 point
(f) Plot qualitatively the solution $u(t)$ for $t > -1$ for the cases in (d) and (e) in the space below. Pay particular attention to the moment around $t = 0$ and to the situation after a very long time.

/2 points

(g) assume that $\tau \gg 1$ (e.g. $\tau = 5$). Assume that we have applied for a long time in the past a negative (hyperpolarizing) current $I = -40$. Draw the nullclines again for this situation in the space below.

What happens if we stop the hyperpolarizing current at $t = 0$ so that $I = 0$ for $t \geq 0$? Explain in words the evolution of the voltage after the current has been stopped and draw a sketch of the trajectory $u(t)$ for $t \geq -1$

/3 points
QUESTION 3: STOCHASTIC MODEL: POISSON PROCESS  (9 points)

We consider a linear neuron model under stochastic spike arrival. The neuron receives input at a rate \( r \) at a synapse with weight \( w \). Each input causes a postsynaptic potential \( \alpha(s) = 2 - s \) for \( 0 < s < 2 \) and zero elsewhere. The total membrane potential is

\[
u(t) = \sum_{t'} w \alpha(t - t') + u_0
\]

The sum is over all spike times arriving at the synapse.

(a) Suppose a fixed value of \( u_0 \). What is the mean membrane potential?
\( \langle u \rangle = \bar{u} = \dot{\ldots} \)

(b) What is the variance of the membrane potential?
\( \langle (u - \bar{u})^2 \rangle = \dot{\ldots} \)

/ 2 points

/ 2 points
(c) Suppose now that because of external input the reference \( u_0 \) is periodically modulated at a period of \( T = 20 \) (e.g., \( T = 20ms \), but you can do the calculation unit-free)

\[
u_0(t) = \Delta u \sin(2\pi t/T)
\]

Stochastic input is the same as before and arrives at a constant rate of \( r = 2 \) (e.g., \( r = 2kHz \)). Sketch 2 sample trajectories of the potential \( u(t) \) corresponding to 2 repetitions of the experiment in the space here. Indicate in your sketch \( u_0(t) \) as well as the mean and the variance. Pay attention to the relation between \( r \) and \( 1/T \).

(d) Suppose that the neuron fires whenever the membrane potential \( u(t) \) hits the threshold \( \theta \) from below. Starting from the figure in (c), choose three different values of theta that lead to qualitatively different firing behavior, and discuss the results. Use 1 sentence for each of the three cases You can use the space below if you want to make additional sketches.

case 1: .......................................................... ..........................................................

case 2: .......................................................... ..........................................................

case 3: .......................................................... ..........................................................

/2 points

/3 points
QUESTION 4: DYNAMICS OF HOPFIELD MODEL (10 points)

Consider a network of $N = 20000$ neurons that has stored 4 patterns

$\xi^1 = \{\xi_1^1, \ldots, \xi_N^1\}$

$\xi^2 = \{\xi_1^2, \ldots, \xi_N^2\}$

$\xi^3 = \{\xi_1^3, \ldots, \xi_N^3\}$

$\xi^4 = \{\xi_1^4, \ldots, \xi_N^4\}$

Using the synaptic update rule $w_{ij} = (J/N) \sum_{\mu} \xi_i^\mu \xi_j^\mu$ where $J > 0$ is a parameter. Each pattern has values $\xi^\mu_i = \pm 1$ so that exactly 50 percent of neurons in a pattern have $\xi^\mu_i = +1$.

Assume stochastic dynamics: neurons receive an input $h_i(t) = \sum_j w_{ij} S_j(t)$ where $S_j(t) = \pm 1$ is the state of neuron $j$. Neurons update their state

$$\text{Prob}\{S_i(t+1) = +1|h_i(t)\} = 0.5[1 + g(h_i(t))] \tag{6}$$

where $g$ is an odd and monotonically increasing function: $g(h) = 2h$ for $|h| < 0.5$ and $g(h) = 1$ for $h \geq 0.5$ and $g(h) = -1$ for $h \leq -0.5$.

(a) Rewrite the righthand-side of equation (6) by introducing an overlap

$m^\mu(t) = (1/N) \sum_j \xi_j^\mu S_j(t)$.

/1 point

(b) What is the significance of the overlap? Describe its meaning in one sentence; give examples if necessary.

/1 point

(c) Assume that the four patterns are orthogonal, i.e., $\sum_i \xi_i^\mu \xi_i^\nu = 0$ if $\mu \neq \nu$. Assume that the overlap with pattern 4 at $t = 0$ has a value of 0.3 and $m^\mu(0) = 0$ for all other patterns.

Suppose that neuron $i$ is a neuron with $\xi_i^4 = -1$.

What is the probability that neuron $i$ fires in time step 1? Give the formula for arbitrary $J$ and evaluate then for $J = 1$.

/2 points
(d) For the same assumptions as in (c), what is the expected overlap for $< m^4(t) >$ after
the first time step.

$< m^4(1) > =$

/ 2 points

(e) For the same assumptions as in (c) and (d), write the evolution of the overlap for
$m^4(t)$ for an arbitrary time step and arbitrary $J$. Assume that $N$ is large ($N \to \infty$).
Because of the orthogonality of the patterns, you may also assume that $m^\mu(t) = 0$ for
$\mu \neq 4$.

$m^4(t + 1) =$

/ 1 point

(f) Can you relate the evolution of the overlap to the more general picture of mean-field
analysis of coupled networks?

/ 2 points
**Question 5: Fokker-Planck Equation**

For a population of integrate-and-fire neurons the continuity equation reads

\[ \tau \frac{\partial}{\partial t} p(u, t) = -\frac{\partial}{\partial u} J(u, t) \]  

(7)

(a) What is the meaning of the term \( p(u, t) \). Give the name and mathematical definition. Describe its significance of \( p \) in your own words.

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/1 point

(b) What is the meaning of the term \( J(u, t) \)? Give the name or definition. Describe the significance of \( J \) in your own words.

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/1 point

(c) A professor who sometimes makes mistakes on the blackboard writes

\[ A(t) = J(\vartheta, t) \]

Is this formula correct? Justify your answer.

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/2 points

(d) How can we include the reset of integrate-and-fire models in the continuity equation above?

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/2 points