Neural Networks and Biological Modeling

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QUESTION SET 1

Exercise 1: Passive Membrane

The voltage across a passive membrane can be described by the equation

\[ \tau \frac{du}{dt} = -(u - u_{\text{rest}}) + RI(t). \]  \hspace{1cm} (1)

1.1 Step current

Consider a current \( I(t) = 0 \) for \( t < t_0 \) and \( I(t) = I_0 \) for \( t > t_0 \). Calculate the voltage \( u(t) \), given that the neuron is at rest at time \( t_0 \). (Hint: Instead of solving the differential equation explicitly, try to construct the response to the step current along the lines: What is the value of \( u(t) \) for \( t \leq t_0 \)? What is the asymptotic value of \( u(t) \) for \( t \gg t_0 \)? What is the functional form and time scale of the transition?)

1.2 Pulse current

Consider a current pulse

\[ I(t) = \begin{cases} 0 & \text{for } t < t_0 \text{ and } t > t_0 + \Delta \\ q/\Delta & \text{for } t \geq t_0 \text{ and } t < t_0 + \Delta, \end{cases} \]  \hspace{1cm} (2)

where \( \Delta \) is a short time and \( q \) is the total electrical charge.

Consider first \( \Delta = 0.1\tau \), and then \( \Delta = 0.05\tau \), \( \Delta = 0.025\tau \). Sketch the input current pulse and the voltage response. What happens in the limit \( \Delta \to 0 \)? (Hint: Use \( e^{-x} \approx 1 - x \) for \( x \ll 1 \).)

1.3 Delta function

The Dirac delta function can be defined by the limit of a short pulse:

\[ \delta(t - t_0) = \lim_{\Delta \to 0} f_\Delta(t) \quad \text{where} \quad f_\Delta(t) = \begin{cases} 1/\Delta & \text{for } t_0 \leq t < t_0 + \Delta \\ 0 & \text{otherwise}. \end{cases} \]  \hspace{1cm} (3)

Convince yourself that the integral \( \int_{t_1}^{t_2} \delta(t - t_0) \, dt \) is equal to one if \( t_1 \leq t_0 < t_2 \) and vanishes otherwise.

Express \( I(t) \) in Eq. 1 using the \( \delta \)-function for the case that an extremely short current pulse arrives at time \( t_f \). Pay attention to the units!

1.4 General solution

Assuming that before a given time \( t_0 \) the current is null and the membrane potential is at rest, derive the general solution to Eq. (1) for arbitrary \( I(t) \).
Exercise 2: Integrate-and-fire model

Consider the model of Eq. (1) with a threshold at $u = \theta > u_{\text{rest}}$. If the membrane potential reaches the threshold, the neuron is said to fire and the membrane potential is reset to $u_{\text{rest}}$. The injected current is a step of magnitude $I_0$:

$$I(t) = \begin{cases} 
0 & t \leq t_0 \\
I_0 & t > t_0 
\end{cases}$$

2.1 What is the minimal current to reach the threshold, assuming $u(t = 0) = u_{\text{rest}}$?

2.2 At what time will the voltage first reach the threshold?

2.3 Calculate the firing frequency $f$ as a function of $I_0$.

The function $g(I_0)$ which gives the firing frequency as a function of the constant applied current is called gain function.

Exercise 3: Integrate-and-fire models

The general form of an integrate-and-fire model is

$$\frac{du}{dt} = F(u) + \frac{RI(t)}{\tau}$$

(4)

where $F(u)$ is an appropriate function and $I(t)$ is the injected current. Three popular choices for the function $F$ are the following (see Fig. 1):

- **Leaky integrate-and-fire** $F(u) = -\frac{u - u_{\text{rest}}}{\tau}$
- **Quadratic integrate-and-fire** $F(u) = k \frac{(u - u_{\text{rest}})(u - u_{\text{th}})}{\tau}$
- **Exponential integrate-and-fire** $F(u) = -\frac{(u - u_{\text{rest}}) + \Delta e^{\frac{u - u_{\text{th}}}{\Delta \tau}}}{\tau}$

3.1 Identify the resting potential $u_{\text{rest}}$ and the spike threshold $u_{\text{th}}$ in Fig. 1.

3.2 Consider three different values $u_1$, $u_2$, and $u_3$ for the voltage such that (i) $u_1$ is below $u_{\text{rest}}$ (the resting potential), (ii) $u_2$ is between $u_{\text{rest}}$ and $u_{\text{th}}$ (the spike threshold), and (iii) $u_3$ is above $u_{\text{th}}$ (see Fig. 1). For the three models described above, determine qualitatively the evolution of $u(t)$ when started at $u_1$, $u_2$, and $u_3$, assuming that the external input $I(t) \equiv 0$.

- For $u(t = 0) = u_1$, the voltage increases/decreases slowly/rapidly.
- For $u(t = 0) = u_2$, ............................
- For $u(t = 0) = u_3$, ............................

3.3 Why is $u_{\text{rest}}$ called the resting potential? What is the role of $u_{\text{th}}$?

3.4 Consider the two voltage traces shown in Fig. 2(b) (top) in response to a step current (bottom). Using the graphs in Fig. 2(a), determine which of the two models was used to generate each trace.
Figure 1: Sketch of the function $F(u)$ for three popular integrate-and-fire models.

Figure 2: Left: Right-hand side of Eq. 4 for the quadratic and exponential integrate-and-fire models if a constant input current $I(t) > 0$ is applied. Lower right: Trace of the injected current. Upper right: Voltage trace of the two models (EIF and QIF).