Exercise 1: Continuous population model

We study a system with lateral connection \( w(x - y) \) given by:
\[
\tau \frac{\partial h(x, t)}{\partial t} = -h(x, t) + \int w(x - y)F[h(y, t)]dy + I_{\text{ext}}(x, t),
\]
(1)

where \( F[h(x, t)] = A(x, t) \) is the population’s activity at the point \( x \) at time \( t \).

1.1 Show that, for a constant current \( I_{\text{ext}} \), the homogeneous stationary solution \( h(x, t) = h_0 \) leads to a constant activity \( A_0 \) given by:
\[
A_0 = F(h_0) = \frac{h_0 - I_{\text{ext}}}{\bar{w}},
\]
with \( \bar{w} = \int w(x - y)dy \).

1.2 We set \( h(x, t) = h_0 + \Delta h(x, t) \) where \( \Delta h \) is a small perturbation. Linearize the equation (1) around \( h_0 \), solve the Fourier transformed equation and obtain \( \Delta h = \int g(k)dk \) where
\[
g(k) = C(k)e^{ikx}e^{-\kappa(k)t/\tau}.
\]
Identify the function \( \kappa \). For which values of \( k \) do we get \( \kappa < 0 \) ?

1.3 Consider:
\[
w(z) = \frac{\sigma_2 e^{-z^2/(2\sigma_1^2)} - \sigma_1 e^{-z^2/(2\sigma_2^2)}}{\sigma_2 - \sigma_1},
\]
with \( \sigma_1 = 1 \) and \( \sigma_2 = 10 \). Sketch the qualitative behaviour of \( w(z) \) and
\[
\int w(z) \cos(kz)dz.
\]
Determine graphically the stability condition.

Exercise 2: Stationary state in a network with lateral connections

Consider a neural network with lateral connections represented in figure 1: the interaction is locally excitatory and long range inhibitory:
\[
w(x, x') = \begin{cases} 
1 & |x - x'| \leq \sigma \\
-b & |x - x'| > \sigma 
\end{cases}
\]
(2)
Therefore \( \sigma \) corresponds to the range of the excitatory connections. The activity \( A \) of a neuron at position \( x \) is given by:
\[
A(x) = F[h(x)],
\]
(3)
where \( h(x) \) is the total potential of the neuron at position \( x \), defined as:
\[
h(x) = \int w(x, x')A(x')dx' + I_{\text{ext}}(x).
\]
(4)
The function $F(h)$ is a simple threshold function:

$$F(h) = \begin{cases} 
1 & h > \theta \\
0 & h \leq \theta 
\end{cases}$$  

(5)

In this exercise we do not add any external input i.e. $I_{ext}(x) = 0$. The aim of the exercise is to find the neural activity $A(x)$. In order to do so, we assume that $A(x)$ may have a rectangular shape (of dimensions $2d \times 1$, as shown in figure 1) and we prove this assumption with the following passages.

2.1 Consider a point at location $x_0$ close to $x = 2d$ and calculate its input potential, assuming that $d > \sigma$.

(Hint: there is excitatory input from the right and there is excitatory and inhibitory input from the left).

2.2 Exploit that at $x_0 = 2d$ we must have $h(x_0) = \Theta$. Why? Calculate $d$.

2.3 Convince yourselves that the bump of size $2d$ is therefore a solution for the activity $A(x)$ and discuss its properties.

Figure 1: Spatial structure of the network.