1. Two more cross-sections

Recall here two useful formulas for the differential and total cross-sections in the first Born approximation that we derived in the previous exercises:

\[
\frac{d\sigma}{d\Omega} = |f|^2 = \left| 2m \int_0^\infty dr V(r) \frac{\sin qr}{q} \right|^2,  \tag{1}
\]

\[
\sigma = \int_0^{2\rho} |f|^2 \frac{2\pi q dq}{p^2},  \tag{2}
\]

where \( q \) is a momentum transfer, \( q = 2\rho \sin \frac{\theta}{2} \).

1. For the potential

\[
V(r) = V_0 e^{-\frac{r^2}{a^2}}  \tag{3}
\]

the formula (1) gives,

\[
\frac{d\sigma}{d\Omega} = \left| 2m V_0 \int_0^\infty dr \sin qr \cdot r e^{-\frac{r^2}{a^2}} \right|^2 = \frac{\pi a^2}{4} (mV_0a^2)^2 e^{-\frac{2q^2}{a^2}},  \tag{4}
\]

and the total cross-section reads as follows,

\[
\sigma = (mV_0a^2)^2 \frac{\pi a^2}{4} \int_0^{2\rho} e^{-\frac{2q^2}{a^2}} \frac{2\pi q dq}{p^2} = \frac{\pi^2}{2p^2} (mV_0a^2)^2 \left(1 - e^{-2\rho^2 a^2}\right).  \tag{5}
\]

Now turn to the potential

\[
V(r) = -e^2 \left(\frac{1}{r} + \frac{1}{a}\right) e^{-\frac{r^2}{a^2}},  \tag{6}
\]

for which the direct calculation leads to

\[
\frac{d\sigma}{d\Omega} = \left[ \frac{e^2}{q} \int_0^\infty dr \left(\frac{1}{r} + \frac{1}{a}\right) e^{-\frac{2r^2}{a^2}} r \sin qr dr \right]^2 = \left[ e^2 a^2 (a^2q^2 + 8) \right]^2 \frac{(mV_0a^2)^2}{(a^2q^2 + 4a^2)^2},  \tag{7}
\]

and

\[
\sigma = e^4 a^4 \int_0^{2\rho} \frac{(a^2q^2 + 8)^2 2\pi q dq}{(a^2q^2 + 4a^2)^4 p^2} = \pi e^4 a^4 \frac{12 + 18a^2p^2 + 7a^4p^4}{12(1 + a^2p^2)^3}.  \tag{8}
\]

2. The applicability conditions of the relations (1) and (2) can be straightforwardly derived from

\[
\frac{m}{p} \left| \int_0^\infty dr V(r)(1 - e^{2ipr}) \right| \ll 1.  \tag{9}
\]
For the first potential we have

\[ pa \ll 1 : \left| \frac{m V_0}{p} \int_0^\infty dr pr e^{-\frac{r^2}{2a^2}} \right| \sim mV_0a^2 \ll 1 ; \]

\[ pa \gg 1 : \left| \frac{m V_0}{p} \int_0^\infty dr \sin pr \cdot e^{-\frac{r^2}{2a^2}} \right| \sim \frac{mV_0a}{p} e^{-\frac{a^2}{p^2}} \ll 1 . \]  

(10)

And for the second potential

\[ pa \ll 1 : \left| \frac{m e^2}{p} \int_0^\infty dr pr \left( \frac{1}{r} + \frac{1}{a} \right) e^{-\frac{r^2}{2a^2}} \right| \sim e^2ma \ll 1 ; \]

\[ pa \gg 1 : \left| \frac{m e^2}{p} \int_0^\infty dr \sin pr \left( \frac{1}{r} + \frac{1}{a} \right) e^{-\frac{r^2}{2a^2}} \right| \sim \frac{e^2m}{p} \ll 1 . \]  

(11)

2. Solution to the Dirac equation

1. Substituting the ansatz \( \Psi_D = e^{i\bar{\Psi} \cdot \vec{\sigma} - i\omega_P t} u_P \) into the Dirac equation,

\[ i \frac{\partial \Psi_D}{\partial t} = H_D \Psi_D , \quad H_D = \sum_{i=1}^3 \alpha_i p_i + \beta m , \quad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} , \quad \beta = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} , \]  

we arrive at the equation

\[ (\alpha p_i + \beta m)u_P = \omega_P u_P . \]  

(13)

2. Eq. (13) is a homogeneous system of four linear equations, schematically \( Au_P = 0 \), where

\[ A = \begin{pmatrix} m - \omega_P & p_i \sigma_i \\ p_i \sigma_i & -m - \omega_P \end{pmatrix} . \]  

(14)

A non-trivial solution to this system exists if and only if the determinant of \( A \) vanishes. That is,

\[ \det A = (m^2 + p_1^2 + p_2^2 + p_3^2 - \omega_P^2)^2 = 0 , \]  

from which it follows that

\[ \omega_P = \pm \sqrt{m^2 + p^2} . \]  

(15)

3. Regarding \( u_P \) as \((\phi_P , \chi_P)^T\), we have

\[ p_i \sigma_i \phi_P = (\omega_P + m)\chi_P , \]

\[ p_i \sigma_i \chi_P = (\omega_P - m)\phi_P . \]  

(17)

Hence, the general solution of the Dirac equation is \( u_P = (\phi_P , (p_i \sigma_i)^{-1}(\omega_P - m)\phi_P)^T \), where \( \phi_P \) is arbitrary.

4. In the non-relativistic limit \( \omega_P = m \), hence \( \chi_P = 0 \), and the solution becomes \( u_P = (\phi_P , 0)^T \).
3. Properties of the Dirac equation

1. The transformation
\[ \alpha_i' = U \alpha_i U^{-1} , \quad \beta' = U \beta U^{-1} \] (18)
does not spoil the properties of \( \alpha_i, \beta \). Indeed, their commutation relations remain unchanged, since, for example,
\[ \alpha_i' \alpha_j' + \alpha_j' \alpha_i' = U(\alpha_i U^{-1} U \alpha_j + \alpha_j U^{-1} U \alpha_i) U^{-1} = 2UU^{-1} \delta_{ij} = 2 \delta_{ij} . \] (19)
Moreover,
\[ \alpha^\dagger_i' = (U \alpha_i U^{-1})^\dagger = (U^{-1})^\dagger \alpha^\dagger_i U = U \alpha^\dagger_i U^{-1} = \alpha_i^\dagger , \] (20)
and similarly \( \beta^\dagger = \beta' \), since the matrix \( U \) is unitary. Hence, hermiticity is also preserved. Finally,
\[ \text{Tr } \alpha_i' = \text{Tr } (U \alpha_i U^{-1}) = \text{Tr } (\alpha_i U^{-1} U) = \text{Tr } \alpha_i = 0 , \] (21)
and similarly \( \text{Tr } \beta' = 0 \).

2. One can choose
\[ U = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} . \] (22)
Then,
\[ \alpha_i' = \begin{pmatrix} -\sigma_i \\ 0 \end{pmatrix} , \quad \beta' = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} . \] (23)

3. In the massless limit, \( m = 0 \), the Dirac equation becomes
\[ i \frac{\partial \Psi_D}{\partial t} = H_D \Psi_D , \quad H_D = \alpha_i' p_i = \begin{pmatrix} -\sigma_i p_i \\ 0 \\ 0 & \sigma_i p_i \end{pmatrix} . \] (24)
By taking \( \Psi_D = (\phi, \chi)^T \), the equation above is spitted into two independent equations:
\[ (i \partial_t + \sigma_i p_i) \phi = 0 \]
\[ (i \partial_t - \sigma_i p_i) \chi = 0 . \] (25)
These equations are called Weyl equations.