Exercice 13.1. Let $M$ be an oriented manifold with boundary. Show that $\partial M$ has a natural orientation.

Solution 13.1. Let $F : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^n$ be a diffeomorphism. which sends $U \cap \mathbb{R}^{n-1} \times \{0\} \to \mathbb{R}^{n-1} \times 0$. Then let $F_0 := F|_{\mathbb{R}^{n-1} \times 0}$. It follows that $\partial_0 F^n = 0$ for any $q \in \mathbb{R}^{n-1} \times \{0\}$. So it follows that $\det (\partial_0 q F^n) = \det DF_q(q)$. Finally $\partial_0 q$ cannot be negative, as then $F(U)$ would not be contained in the upper half plane. Now let $\varphi : U \to \mathbb{R}^n$ and $\psi : V \to \mathbb{R}^n$, and let $\varphi_0 = \psi|_{\partial M \cap U}$. Let $\Phi_{\partial UV}$ be the restriction of $\Phi_{UV}$ to $\varphi(\partial M \cap U \cap V)$. Then by the previous

$$0 < \det D\Phi_{UV}(\varphi_0(q)) = \det \varphi_0(\varphi(q)) \Phi_{UV} \det D\Phi_{\partial UV}(\varphi_0(q)).$$

But as before $\partial_n \Phi_{UV} \geq 0$, and so $\partial_n \Phi_{UV}$ and $\det D\Phi_{\partial UV}$ are strictly positive. Consequently $\partial M$ has a transition maps with positive Jacobian determinant ($\det D\Phi_{\partial UV}$).

Exercice 13.2. Show that for $k$-form $\alpha$ and $n - k - 1$ form $\beta$

$$\int_M \alpha \wedge d\beta = (-1)^k \left( \int_{\partial M} \alpha \wedge \beta - \int_M d\alpha \wedge \beta \right).$$

This is the integration by parts formula for differential forms.

Solution 13.2. Recall that $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{k(n-k-1)} \alpha \wedge d\beta$. If we apply Stokes’ theorem to both sides, we get

$$\int_{\partial M} \alpha \wedge \beta = \int_M d\alpha \wedge \beta + (-1)^k \int_M \alpha \wedge d\beta,$$

which yields the result after rearranging and multiplying by $(-1)^k$

Exercice 13.3. Let $M$ be a compact orientable $n$-dimensional manifold without boundary. Show that $H^n(M) \neq 0$.

Solution 13.3. Let $\eta$ be a non-negative plateau function whose support is in a coordinate chart $\varphi : U \to \mathbb{R}^n$. Then let $\omega = \eta dx^1 \wedge \cdots \wedge dx^n$, where $x^i$ are the coordinate functions of $\varphi$. The form $\omega$ can be extended to a smooth form on $M$ by setting it to be zero outside of $U$. Then

$$\int_M \omega = \int_{\mathbb{R}^n} \eta \circ \varphi^{-1}(q) \, dq > 0.$$

But if $\omega = d\xi$ then

$$\int_M d\xi = \int_{\partial M} \xi = \int_{\partial \varphi} \xi = 0.$$

Consequently $\omega$ is not exact and it is closed because it is an $n$-form, so $H^n(M)$ is non-zero.

Exercice 13.4. Consider the form $\omega \in \Omega^{n-1}(\mathbb{R}^n \setminus \{0\})$

$$\omega = \sum_{i=1}^n (-1)^{i+1} \frac{x^i}{r^n} dx^{N\setminus\{i\}},$$

where

$$r = \sqrt{\sum_i (x^i)^2},$$

and $N = \{1, \ldots, n\}$. Show that $d\omega = 0$ Show using Stokes’ theorem that $H^{n-1}(\mathbb{R}^n \setminus \{0\})$ is non-zero.
Hint. First recall that if $\omega = d\xi$ then $i^*(\omega) = i^*(d\xi) = di^*(d\xi)$, where $i : S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ is the inclusion map. In words: if the form is exact then it’s restriction is exact. Now let

$$\tilde{\omega} = \sum_i (-1)^{i+1} x^i dx_N \setminus \{i\}.$$  

Then $i^*(\omega) = i^*(\tilde{\omega})$ because $r \circ i \equiv 1$. Apply Stokes’ theorem to $\tilde{\omega}$.

Solution 13.4. First

$$d\omega = \sum_i (-1)^{i+1} \frac{1}{p^n} dx^i \wedge dx_N \setminus \{i\} - (-1)^{i+1} (n) \frac{x^i}{p^{n+2}} dr \wedge dx_N \setminus \{i\}$$

$$= \frac{n}{p^n} dx^N - n \sum_{i=1}^n (-1)^{n+1} (x^i)^2 \frac{x^i}{p^{n+2}} dx^i \wedge dx_N \setminus \{i\}$$

$$= \left( \frac{n}{p^n} - \frac{n x^2}{p^{n+2}} \right) dx^N = 0.$$  

Now consider the the restriction of $\omega$ to $S^{n-1}$. If we can show the integral is non-zero, then we are done. But for $r = 1$ this is the integral of the form $\Omega = \sum_i (-1)^{i+1} x^i dx_N \setminus \{i\}$:

$$\int_{S^{n-1}} \omega = \int_{S^{n-1}} \Omega = \int_{\partial B^n} \Omega = \int_{B^n} d\Omega.$$  

But $d\Omega = ndx^N$, so $\int_{S^{n-1}} \omega = n \text{Vol}(B^n) \neq 0$. And so $\omega$ cannot be exact.

Exercice 13.5. Recall Green’s theorem

$$\iint_{\Omega} \frac{\partial F^1}{\partial x^2} - \frac{\partial F^2}{\partial x^1} \, dx^1 \, dx^2 = \oint_{\partial \Omega} F \cdot dl$$

Gauss’ theorem

$$\iiint_{\Omega} \nabla \cdot F \, dx^1 \, dx^2 \, dx^3 = \iint_{\partial \Omega} F \cdot \nu dS$$

and Stoke’s theorem (from vector calculus)

$$\iint_{\Sigma} \nabla \times F \cdot \nu dS = \oint_{\partial \Sigma} F \cdot dl.$$  

See Analyse avancée pour ingénieurs Chap. 4,6,7. Identify how each of these is Stokes’ theorem for manifolds by identifying $F$ with a differential form, what are the manifolds of integration and what are the boundaries. Calculate $dF$ in each case.

Solution 13.5. For Green’s theorem, $\Omega$ is a two dimensional manifold and $F$ is a one form $F = F^1 dx^1 + F^2 dx^2$.

Then

$$dF = \left( - \frac{\partial F^1}{\partial x^2} + \frac{\partial F^2}{\partial x^1} \right) \, dx^1 \wedge dx^2.$$
And so this is
\[ \int_{\Omega} dF = \int_{\partial\Omega} F \]
for \( F \) a one-form and \( \Omega \) a 2-manifold. For Gauss’ theorem
\[ F = F^1 dx^2 \wedge dx^3 - F^2 dx^1 \wedge dx^3 + F^3 dx^1 \wedge dx^2. \]
From Exercise 12.5, we recall that \( dF = \nabla \cdot F \), and so Gauss’ theorem corresponds to
\[ \int_{\Omega} dF = \int_{\partial\Omega} F, \]
for \( F \) a 2 form and \( \Omega \) a 3-manifold.
For Stokes’ theorem let
\[ F = F^1 dx^1 + F^2 dx^2 + F^3 dx^3, \]
then \( dF \) corresponds to \( \nabla \times F \), and it corresponds to Stokes’ theorem on a 2-manifold for a one form \( F \).

**Exercise 13.6.** Let \( \alpha \) and \( \beta \) be closed forms. Show that \((\alpha \wedge \beta - (\alpha + d\xi) \wedge (\beta + d\chi))\) is exact. Deduce that the wedge product defines a bilinear map
\[ \cdot \wedge \cdot : H^k(M) \times H^l(M) \to H^{k+l}(M). \]
This is called the *cup product* and it makes \( H^*(M) \) an algebra. Consequently deduce that if \([\alpha \wedge \beta]\) is not zero in cohomology (i.e. \( \alpha \wedge \beta \neq 0 \) in the cohomology), then neither \( \alpha \) nor \( \beta \) is exact.

Consider \( T^n = \mathbb{R}^n / \mathbb{Z}^n \) (i.e. \( q \sim p \) if \( q - p \in \mathbb{Z}^n \)). Consider the chart \( \varphi : \mathbb{R}^n \setminus \bigcup \{ q \in \mathbb{R}^n \mid q^i \in \mathbb{Z} \}, q \mapsto (q^1 \mod 1, \ldots, q^n \mod 1) \). Let \( x^i \) denote the coordinate functions of this chart. As in exercise 11.5 one can deduce that this one-form extends to a smooth one form, which we call \( \theta^i \) (you do not have to). Deduce that \( \theta^i \wedge \cdots \wedge \theta^n \) is non-zero in cohomology. Deduce that \( \dim H^k(T^n) \geq \binom{n}{k} \).

**Solution 13.6.** We calculate
\[ \alpha \wedge \beta - (\alpha + d\xi) \wedge (\beta + d\chi) = -\alpha \wedge d\chi - d\xi \wedge \beta - d\xi \wedge d\chi = (-1)^{k+1} d(\alpha \wedge \chi) - d(\xi \wedge \beta) - d(\xi \wedge d\chi) = -d((-1)^k \alpha \wedge \chi + \xi \wedge \beta + \xi \wedge d\chi). \]
Hence the map \([\alpha], [\beta] \mapsto [\alpha \wedge \beta]\) is well defined. It is immediately bilinear because \( \alpha, \beta \mapsto \alpha \wedge \beta \) is bilinear. If \([\alpha \wedge \beta]\) is non-zero, then neither \([\alpha]\) nor \([\beta]\) can be zero, as then it is cohomologically equivalent to \( 0 \) or \( 0 \wedge \beta \) which are both 0.

We need to show that \( \int_{T^n} \theta^1 \wedge \cdots \wedge \theta^n \neq 0 \). Then the closed form \( \theta^I \) are linearly independent and \( \theta^I \wedge \theta^{N \setminus I} = \pm \theta^N \) is not exact, so neither \( \theta^I \) nor \( \theta^J \) are exact. There are \( \binom{n}{k} \) subsets \( I \) of \( N \) of size \( k \), so this is a lower bound for the dimension. To see that \( \int_{T^n} \theta^N \) is non-zero, note that the complement of \( U \) the domain of \( \varphi \) is a union of \( n-1 \) dimensional manifolds. So if we consider \( \int_{T^n} \theta^N = \int_{T^n} \sum \eta_a \theta^N \) for some partition of unity, we can integrate over just \( U \). That is for a domain in \( \mathbb{R}^n, \Omega \), and an \( n-1 \) dimensional subset set \( \Sigma \)
\[ \int_{\Omega \setminus \Sigma} f(x) dx = \int_{\Omega} f(x) dx. \]
Consequently

\[
\int_{T^n} \theta^N = \int_{T^n} \sum_{\alpha} \eta_{\alpha} \theta^N = \int_{\varphi_a(U_a)} \varphi_{\alpha}^{-1}(\eta_{\alpha} \theta^\alpha) \\
= \sum_{\alpha} \int_{\varphi_a(U_a \cap U)} \varphi_{\alpha}^{-1}(\eta_{\alpha} \theta^N) \\
= \sum_{\alpha} \int_{\varphi(U \cap U_a)} \varphi_{\alpha}^{-1}(\eta_{\alpha} \theta^N) \\
= \int_{[0,1]^n} dx \\
= 1.
\]

**Exercice 13.7. (Les équations de Maxwell)** On considère l’espace de Lorentz–Minkowski, qui n’est rien d’autre que \(\mathbb{R}^4\) muni de coordonnées \(x, y, z, t\) où \(x, y, z\) représentent les coordonnées “spatiales” et \(t\) la coordonnée “temporelle”. Les champs électrique \(E\) et magnétique \(B\) sont des champs de vecteurs définis sur \(\mathbb{R}^3\) qui dépendent du temps et les équations de Maxwell dans le vide s’écrivent

\[
\text{rot}(E) = - \frac{\partial B}{\partial t} \quad \text{div}(B) = 0 \\
\text{rot}(B) = \frac{\partial E}{\partial t} \quad \text{div}(E) = 0.
\]

Le but de cet exercice est de reformuler ces équations dans le langage des formes différentielles. On définit l’opérateur de Hodge lorentzien \(* : \Omega^2(\mathbb{R}^4) \rightarrow \Omega^2(\mathbb{R}^4)\) par

\[
* (dx \wedge dt) = dy \wedge dz, \quad * (dy \wedge dt) = dz \wedge dx, \quad * (dz \wedge dt) = dx \wedge dy,
\]

et

\[
* (dy \wedge dz) = - dx \wedge dt, \quad * (dz \wedge dx) = - dy \wedge dt, \quad * (dx \wedge dy) = - dz \wedge dt,
\]

Maintenant on introduit le 2-forme \(F \in \Omega^2(\mathbb{R}^4)\) :

\[
F = E_x dx \wedge dt + E_y dy \wedge dt + E_z dz \wedge dt + B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy.
\]

Ce tenseur s’appelle le “champ électromagnétique”. Montrer que les équations de Maxwell (dans le vide) s’écrivent

\[
dF = 0, \quad d * F = 0
\]

**Solution 13.7.** Le calcul de \(dF = 0\) mène aux équations : \(\text{rot}(E) = - \frac{\partial B}{\partial t}\) et \(\text{div}(B) = 0\). et le calcul de \(d * F = 0\) donne les deux autres équations.