1. Optical theorem and the Born approximation

The important observation is that the total cross section, calculated in the first Born approximation, is actually of the second order in the small parameter $\lambda$. That is,

$$\int d\Omega |f^{(1)}|^2 = \sigma^{(2)}. \quad (1)$$

To check the validity of the optical theorem, one should compare this result with the amplitude calculated up to the second order in $\lambda$:

$$\sigma^{(2)} = \frac{4\pi}{p} \text{Im} f^{(2)}(\vec{p} - \vec{p}). \quad (2)$$

Recall that

$$f^{(2)}(\vec{p}' \leftarrow \vec{p}) = -(2\pi)^2 m \int d^3 \vec{p}'' |(\vec{p}'|\vec{p}''\rangle\langle\vec{p}''|\vec{p})|^2 \frac{1}{E_p + i\epsilon - E_{p''}}. \quad (3)$$

Let us take the imaginary part of this expression, assuming $\vec{p} = \vec{p}'$ (forward scattering):

$$\text{Im} f^{(2)}(\vec{p}' \leftarrow \vec{p}) = -(2\pi)^2 m \int d^3 \vec{p}'' |\vec{V}(\vec{p}'' - \vec{p})|^2 \text{Im} \frac{1}{E_p + i\epsilon - E_{p''}}, \quad (4)$$

where we used the relation

$$\frac{1}{x \pm i\epsilon} = \mathcal{P} \frac{1}{x} \mp i\pi \delta(x). \quad (5)$$

On the other hand,

$$\sigma^{(2)} = \int d\Omega |f^{(1)}|^2 = 16\pi^4 m^2 \int d\Omega |\vec{V}(\vec{p}' - \vec{p})|^2. \quad (6)$$

Rewrite the angular integral as follows,

$$\int d\Omega = \frac{1}{p^2} \int d^3 \vec{p}' \delta(p' - p). \quad (7)$$

Then Eq.(6) is rewritten as

$$\sigma^{(2)} = \frac{16\pi^4 m^2}{p^2} \int d^3 \vec{p}' \delta(p' - p)|\vec{V}(\vec{p}' - \vec{p})|^2. \quad (8)$$

Finally, the delta-function is given by $\delta(p - p') = p/m \cdot \delta(E_p - E_{p'})$. Comparing Eqs.(4) and (8), we see that

$$\frac{4\pi}{p} \text{Im} f^{(2)}(\vec{p} - \vec{p}) = \sigma^{(2)}. \quad (9)$$
Hence, the optical theorem holds to the first non-trivial order in $\lambda$.

2. Spherical well potential

1. The scattering amplitude in the first Born approximation is written as

$$f^{(1)}(\vec{p}' \leftarrow \vec{p}) = \frac{m}{2\pi} \int d^3\vec{r} V(\vec{r}) e^{-i\vec{q} \cdot \vec{r}},$$

where $\vec{q} = \vec{p}' - \vec{p}$ is a momentum transfer. For a spherical symmetric potential the expression above reduces to

$$f^{(1)}(\vec{p}' \leftarrow \vec{p}) = -2m \int_0^\infty dr V(r) \frac{\sin qr}{q}.$$  \hspace{1cm} (11)

We will investigate the following potential,

$$V(r) = \begin{cases} -V_0, & r < a, \\ 0, & r > a. \end{cases}$$ \hspace{1cm} (12)

Substituting it into Eq.(11), we have

$$f^{(1)} = 2mV_0 \int_0^a dr \frac{\sin qr}{q}.$$ \hspace{1cm} (13)

Computing this integral, we obtain,

$$\frac{d\sigma}{d\Omega} = |f^{(1)}|^2 = 4a^2(mV_0a^2)^2 \frac{(\sin qa - qa \cos qa)^2}{qa^6}.$$ \hspace{1cm} (14)

2. We observe that $q = 2p \sin \theta/2$, and, hence, $d\Omega = d\phi d\cos \theta = 2\pi q dq / p^2$. For the total cross section we, therefore, have

$$\sigma = \int_0^{2p} \frac{2\pi q dq}{p^2} |f^{(1)}|^2.$$ \hspace{1cm} (15)

Straightforward calculation gives,

$$\sigma = \frac{2\pi}{p^2} (mV_0a^2)^2 \left[ 1 - \frac{1}{(2pa)^2} + \frac{\sin 4pa}{(2pa)^3} - \frac{\sin^2 2pa}{(2pa)^4} \right].$$ \hspace{1cm} (16)

3. In the regime $pa \ll 1$, the wave length of the particles is much larger than the size of the potential well. One should expand Eq.(16) up to the first non-trivial order in $pa$. This gives,

$$\sigma = \frac{16\pi a^2}{9} (mV_0a^2)^2.$$ \hspace{1cm} (17)

The cross section does not depend on the momentum of the particles, as expected. Indeed, the scattering amplitude (11) does not depend on $q$ in the limit $q \to 0$.

4. In the opposite limit, $pa \gg 1$, we have

$$\sigma = \frac{2\pi}{p^2} (mV_0a^2)^2.$$ \hspace{1cm} (18)

The cross section goes to zero as the energy of the particles increases.
3. Completeness relation from the Green’s function

The integral of the function $\hat{G}(z) = \frac{1}{z - \hat{H}}$ over the closed contour shown in Fig.(1) is given by the sum of the residues computed at singular points of $\hat{G}(z)$ located inside the contour. The only such points are the poles corresponding to the bound states of the operator $\hat{H}$. Hence, we have

$$\oint_{C} \hat{G}(z) dz = 2\pi i \sum_{n} |n\rangle\langle n|.$$  \hspace{1cm} (19)

On the other hand, one can compute the integral in the l.h.s. of Eq.(19) explicitly. The integral over the circle gives,

$$\int_{\text{circle}} \frac{dz}{z - \hat{H}} = \lim_{R \to \infty} R \int_{0}^{2\pi} \frac{d\phi e^{i\phi}}{Re^{i\phi} - \hat{H}} = \lim_{R \to \infty} \left[ \log(e^{i\phi} - \frac{\hat{H}}{R})_{\phi=2\pi} - \log(e^{i\phi} - \frac{\hat{H}}{R})_{\phi=0} \right] = 2\pi i.$$  \hspace{1cm} (20)

The integral over the branch cut is computed as follows,

$$\int_{\text{branch cut}} \hat{G}(z) dz = \int_{0}^{\infty} dx \int dp |p\rangle\langle p| \left( \frac{1}{x + i\epsilon - E_{p}} - \frac{1}{x - i\epsilon - E_{p}} \right) = -2\pi i \int dp |p\rangle\langle p|.$$

where $|p\rangle$ denote eigenstates of the continuous spectrum of $\hat{H}$, and we used the relation

$$\frac{1}{x \pm i\epsilon} = \mathcal{P} \frac{1}{x} \mp i\pi \delta(x).$$  \hspace{1cm} (22)

Equating the r.h.s. of Eq.(19) to the sum of r.h.s. of Eqs.(20),(21), we obtain

$$\sum_{n} |n\rangle\langle n| + \int dp |p\rangle\langle p| = 1.$$  \hspace{1cm} (23)

Fig. 1 – The contour of integration $C$. Red dots correspond to the energies of the bound states of $\hat{H}$, the red line corresponds to the continuous spectrum.