1. Optical theorem and the Born approximation

Recall that the optical theorem relates the scattering amplitude $f(\vec{p}' \leftarrow \vec{p})$ in the forward direction, $\vec{p}' = \vec{p}$, to the total cross section:

$$\sigma = \frac{4\pi}{p} \text{Im} f(\vec{p} \leftarrow \vec{p}).$$  \hspace{1cm} (1)

1. Assuming $\hat{H} = \hat{H}_0 + \lambda \hat{V}$, $\lambda \ll 1$, check explicitly Eq.(1) to the first non-trivial order in $\lambda$.

2. Spherical well potential

In this exercise we will find an approximate formula for the total cross section of the following spherical symmetric potential:

$$V(r) = \begin{cases} -V_0, & r < a, \\ 0, & r > a. \end{cases}$$  \hspace{1cm} (2)

1. In the first Born approximation, find the differential cross section $d\sigma/d\Omega$ for the potential (2).
2. Find the total cross section $\sigma = \int d\sigma d\Omega$.
   
   *Hint*: Deduce and use the following formula for the solid angle element:

$$d\Omega = \frac{2\pi q dq}{p^2}, \quad q = |\vec{p} - \vec{p}'|.$$  \hspace{1cm} (3)

*Note*: Let us now explore the different regimes of the general formula for $\sigma$. Consider first the low energy limit $pa \ll 1$, in which case the wave length of the particles is much larger than the size of the potential.

3. Expanding $\sigma$ with respect to $pa$, find its form in the low energy limit. Explain the dependence of $\sigma$ on the initial momentum $p$ of the particles in this limit.
4. Write $\sigma$ in the opposite limit of high energy scattering, $pa \gg 1$. What is the dependence on $p$ in this case?

3. Completeness relation from the Green’s function

Let $\hat{G}(z) = \frac{1}{\hat{H} - z}$, where $\hat{H}$ is a Hamiltonian of some system containing a finite amount of bound states $|n\rangle$ with energies $E_n < 0$ and a continuous spectrum $|q\rangle$ starting at $E = 0$. 


1. Prove the completeness relation for the eigenstates of $\hat{H}$,

$$\sum_n |n\rangle\langle n| + \int dq |q\rangle\langle q| = 1,$$

by computing the integral $\oint_C \hat{G}(z) dz$, where the contour $C$ is shown in Fig.(1).

![Fig. 1 – The contour of integration $C$. Red dots correspond to the energies of the bound states of $\hat{H}$, the red line corresponds to the continuous spectrum.](image-url)