1. A delta-function

One has to compute the integral

\[ I = \int dx f(x) \delta(ax^2 + bx + c). \]  

Denote \( g(x) = ax^2 + bx + c \).

— If the equation \( g(x) = 0 \) has no real roots, then the argument of the delta-function is never zero, hence \( I = 0 \).

— Let now the equation \( g(x) = 0 \) have two different real roots \( x_1, x_2 \). Near each of them, the function \( g(x) \) can be written as \( g(x) = g'(x_1)(x - x_1) + O((x-x_1)^2) \). Let \( O_1 \) and \( O_2 \) be small neighborhoods of the points \( x_1 \) and \( x_2 \) correspondingly. The integral \( I \) becomes

\[ I = \int_{O_1} dy \frac{dy}{|g'(x_1 + y)|} f(x_1 + y) \delta(y) + \int_{O_2} dy \frac{dy}{|g'(x_2 + y)|} f(x_2 + y) \delta(y), \]  

where we made a change of variable \( y = x - x_1 \) in the first integral, \( y = x - x_2 \) in the second integral, and used the property of the delta-function,

\[ \delta(\alpha x) = \frac{1}{|\alpha|} \delta(x), \]  

with \( \alpha \) some constant. Taking the integrals, we have

\[ I = \frac{f(x_1)}{|g'(x_1)|} + \frac{f(x_2)}{|g'(x_2)|}. \]  

Finally, \( |g'(x_1)| = |g'(x_2)| = |a(x_1 - x_2)| = \sqrt{b^2 - 4ac} \), and

\[ I = (f(x_1) + f(x_2))(b^2 - 4ac)^{-1/2}. \]  

— Let now \( x_1 = x_2 = x_0 \). Expanding \( g(x) \) around \( x_0 \) and changing the variables as above, we have

\[ I = \int \frac{dy f(x_0 + y)}{|g'(x_0 + y)|} \delta(y) = \begin{cases} \infty, & f'(x_0) \neq 0, \\ \frac{f'(x_0)}{2|a|}, & f(x_0) = 0. \end{cases} \]
2. Green’s function of free particle

1. By definition,
   \[ \hat{G}_0(z) = \frac{1}{z - \hat{H}_0}. \]  
   (7)

   This means that
   \[ \hat{G}_0(z)|\vec{p}\rangle = \frac{1}{z - E_p}|\vec{p}\rangle, \quad E_p = \frac{p^2}{2m}. \]  
   (8)

   Therefore,
   \[ \langle \vec{x}|\hat{G}_0(z)|\vec{x}'\rangle = \int d^3\vec{p}\langle \vec{x}|\hat{G}_0(z)|\vec{p}\rangle\langle \vec{p}|\vec{x}'\rangle = \frac{1}{(2\pi)^3} \int d^3\vec{p} \frac{e^{i\vec{p} \cdot (\vec{x} - \vec{x}')}}{z - E_p}. \]  
   (9)

2. Let us first compute the radial part of the integral:
   \[ \langle \vec{x}|\hat{G}_0(z)|\vec{x}'\rangle = 2\pi \frac{1}{(2\pi)^3} \int_0^\infty dp \frac{p^2}{2m} \int_0^\pi d\theta \sin \theta \frac{e^{i|\vec{p}|(\vec{x} - \vec{x}')\cos \theta}}{z - E_p} = \]  
   \[ -\frac{1}{(2\pi)^2} \int_0^\infty dp \frac{p^2}{2m} \frac{1}{z - E_p} \left( e^{i|\vec{p}|(\vec{x} - \vec{x}')} - e^{-i|\vec{p}|(\vec{x} - \vec{x}')} \right) = \]  
   \[ \frac{im}{2\pi^2|\vec{x} - \vec{x}'|} \int_{-\infty}^\infty dp \frac{p e^{i|\vec{p}|(\vec{x} - \vec{x}')}}{2mz - p^2}. \]  
   (10)

   The last integral can be calculated using the method of residues. To this end, we close the contour of integration in the complex plane as shown in Fig.1. This procedure does not change the value of the integral, since in the upper half-plane the integrand goes to zero exponentially fast as the radius of the semi-circle increases. Now we can apply the method of residues. Recalling that \( z = E + i\epsilon, \epsilon > 0 \), we see that the only pole which contributes to the integral is at \( p = \sqrt{2mz} \). Hence,

   \[ \langle \vec{x}|\hat{G}_0(z)|\vec{x}'\rangle = \frac{im}{2\pi^2|\vec{x} - \vec{x}'|} \int_{-\infty}^\infty dp \frac{p e^{i|\vec{p}|(\vec{x} - \vec{x}')}}{2mz - p^2} = \frac{m e^{i\sqrt{2mz}|\vec{x} - \vec{x}'|}}{2\pi|\vec{x} - \vec{x}'|}. \]  
   (11)

3. The formula (9) tells us that the Fourier transform of the function \( G_0(z, \vec{x}, \vec{x}') = \langle \vec{x}|\hat{G}_0(z)|\vec{x}'\rangle \) is
   \[ G_0(z, \vec{p}, \vec{p}') = \delta(\vec{p} - \vec{p}') \frac{1}{z - E_p}. \]  
   (12)

   Among other things, this means that the momentum of the free particle is a conserved quantity. Let us use this momentum representation for the Green’s function as
follows,
\[
\langle \vec{x}|(z - \hat{H}_0)\hat{G}_0(z)|\vec{x}'\rangle = \int d^3\vec{p}\, d^3\vec{p}'\, d^3\vec{p}'' \langle \vec{x}|\vec{p}\rangle\langle \vec{p}|z - \hat{H}_0|\vec{p}''\rangle\langle \vec{p}''|\hat{G}_0(z)|\vec{p}'\rangle\langle \vec{p}'|\vec{x}\rangle = \int d^3\vec{p}\, d^3\vec{p}'\, d^3\vec{p}'' e^{i\vec{p}\cdot\vec{x}} \delta(\vec{p} - \vec{p}'') (z - E_{p''}) \frac{1}{z - E_p} e^{-i\vec{p}\cdot\vec{x}} = \int d^3\vec{p} e^{i\vec{p}\cdot\vec{x}} e^{-i\vec{p}'\cdot\vec{x}} \delta(\vec{p} - \vec{p}') = \int d^3\vec{p} e^{i\vec{p}\cdot\vec{x}} = \delta(\vec{x} - \vec{x}').
\]

(13)

Fig. 1 – The rule of bypassing the poles.

4. The matrix element $G_0(z, \vec{x},\vec{x}')$, as a function of the complex variable $z$, has a branch cut along the real positive values of $z$. To calculate the difference between the points on the opposite sides of the branch cut, one should continue analytically the function $\sqrt{z}$ from the one side to another. This gives,
\[
G_0(E + ie, \vec{x},\vec{x}') - G_0(E - ie, \vec{x},\vec{x}') = -\frac{m}{2\pi|\vec{x} - \vec{x}'|} \left( e^{i\sqrt{2mE}|\vec{x} - \vec{x}'|} - e^{-i\sqrt{2mE}|\vec{x} - \vec{x}'|} \right) = -\frac{im}{\pi|\vec{x} - \vec{x}'|} \sin \left( \sqrt{2mE}|\vec{x} - \vec{x}'| \right).
\]

(14)

5. For large values of $x = |\vec{x}|$, the modulus $|\vec{x} - \vec{x}'|$ can be expanded as
\[
|\vec{x} - \vec{x}'| = x \sqrt{1 - \frac{2\vec{x} \cdot \vec{x}'}{x^2} + \frac{x'^2}{x^2}} \approx x - \frac{\vec{x} \cdot \vec{x}'}{x}.
\]

(15)

Thus,
\[
G_0(z, \vec{x},\vec{x}') \approx -\frac{m}{2\pi} \exp \left[ i\sqrt{2mE} \left( x - \frac{\vec{x} \cdot \vec{x}'}{x} \right) \right] \frac{x}{x}, \quad x \to \infty.
\]

(16)
3*. Friedel sum rule

1. The eigenstates of the Hamiltonian \( \hat{H} \) form an orthonormal basis of states. This allows us to write

\[
\hat{G}(x + i\epsilon) = \frac{1}{x - \hat{H} + i\epsilon} = \sum_n \frac{|n\rangle\langle n|}{x - E_n + i\epsilon}.
\]  

(17)

Here by \(|n\rangle\) we imply both the bound and scattering states. For the matrix element we have,

\[
G_{nm}(x + i\epsilon) = \langle n|\hat{G}(x + i\epsilon)|m\rangle = \sum_n \frac{\delta_{nm}}{x - E_n + i\epsilon},
\]  

(18)

or

\[
G_{nn}(x + i\epsilon) = \frac{1}{x - E_n + i\epsilon} = \frac{d}{dx} \log G_{nn}(x + i\epsilon) = -\frac{1}{\pi} \frac{d}{dx} \log G_{nn}(x + i\epsilon).
\]  

(19)

Now we turn to the function \( N(x) \). Using the relation

\[
\frac{1}{x + i\epsilon} = -i\pi \delta(x) + P \frac{1}{x},
\]  

(20)

we have,

\[
N(x) = \sum_n \delta(x - E_n) = -\frac{1}{\pi} \sum_n \text{Im} \frac{1}{x - E_n + i\epsilon} = -\frac{1}{\pi} \text{Im} \sum_n G_{nn}(x + i\epsilon).
\]  

(21)

Substitution of the expression (19) then leads to

\[
N(x) = \frac{1}{\pi} \text{Im} \sum_n \frac{d}{dx} \log G_{nn}(x + i\epsilon) = \frac{1}{\pi} \frac{d}{dx} \text{Im} \log \det \hat{G}(x + i\epsilon).
\]  

(22)

2. From Eq.(22) it follows that

\[
N(x) - N_0(x) = \frac{1}{\pi} \frac{d}{dx} \text{Im} \left[ \log \det \hat{G}(x + i\epsilon) - \log \det \hat{G}_0(x + i\epsilon) \right] =
\]

\[
\frac{1}{\pi} \frac{d}{dx} \text{Im} \log \det \hat{G}_0^{-1}(x + i\epsilon).
\]  

(23)

Writing \( \det \hat{G}_0^{-1} \) as \( \det \hat{G}_0^{-1} e^{i\arg \det \hat{G}_0^{-1}} \), we have

\[
N(x) - N_0(x) = \frac{1}{\pi} \frac{d}{dx} \arg \det \left( \hat{G}_0^{-1}(x + i\epsilon) \right).
\]  

(24)