Exercice 8.1. Let $\mathcal{V}$ be a finite dimensional vector space (and hence isomorphic (and homeomorphic) to $\mathbb{R}^n$ for some $n$). Let $V$ be an open subset of $\mathcal{V}$. For every $p \in V$, show that $T_pV$ canonically isomorphic to $V$. We can thus identify $TV$ with $V \times V$.

Exercice 8.2. Let $M \subset \mathbb{R}^n$ be a smooth submanifold, show that $T_pM \subset T_p\mathbb{R}^n$. From the previous exercise $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$. Characterise $TS^{n-1}$ as a subset of $T\mathbb{R}^n$.

Exercice 8.3. Let $N$ and $W$ be submanifolds of $M$ with dimensions $n_w$ and $m$ respectively. Show that for $p \in N$ there is a natural inclusion $T_pN \subset T_pM$ and similarly $q \in W \cap N T_qM = T_qW + T_qN$. Show that if $N$ and $W$ intersect transversely, $\dim(T_pN \cap T_qW) = \dim N + \dim W - \dim M$.

Assume that for every point $y \in W$ there is a a chart $\varphi : U \subset M \to \mathbb{R}^{m-w} \times \mathbb{R}^w$ with $y \in U$ such that $W \cap U = \varphi^{-1}(\{0\} \times \mathbb{R}^w)$. Assume the same for $N$. Show using the implicit function theorem that if $N$ and $W$ intersect transversely, then $N \cap W$ is a smooth manifold with tangent plane given by $T_pN \cap T_qW$.

Hint. Consider charts $\varphi : U \subset M \to \mathbb{R}^{m-w} \times \mathbb{R}^2$ and $\psi : V \subset M \to \mathbb{R}^{m-n} \times \mathbb{R}^n$. Consider the function $F : U \cap V \to \mathbb{R}^{m-w} \times \mathbb{R}^{m-n}$ given by

$$p \mapsto (\Pi_{\mathbb{R}^{m-w}} \circ \varphi(p), \Pi_{\mathbb{R}^{m-n}} \circ \psi(p)),$$

where $\Pi_{\mathbb{R}^{m-w}}$ is the projection onto the first $m - w$ coordinates, and $\Pi_{\mathbb{R}^{m-n}}$ is the projection onto the first $m - n$ coordinates. Consider $F^{-1}(\{0\}) = U \cap V \cap W \cap N$.

Exercice 8.4. Utiliser l’application suivante pour montrer que $O(n)$ est une sous-variété de $GL_n(\mathbb{R})$ :

$$\phi : GL_n(\mathbb{R}) \to \text{Sym}_n(\mathbb{R}), \quad \phi(A) = A \cdot A^T - I_n,$$

où $\text{Sym}_n(\mathbb{R})$ est l’ensemble des matrices symétriques à coefficients dans $\mathbb{R}$. Quelle est sa dimension ? Montrer ensuite que $SO(n)$ est une sous-variété de $O(n)$ et donner sa dimension.