1. Spherical symmetry

1. Consider some potential which admits bound states and is regular at $x = 0$, see Fig.1a. To the right side of the turning point $x = a$ in the semiclassical approximation one can write, $\Psi_{x > a}(x) = \frac{C}{2\sqrt{|p|}} e^{-\frac{i}{\hbar} \int_{a}^{x} |p| dx}$. Analytical continuation to $x < a$ gives, $\Psi_{x < a}(x) = \frac{C}{\sqrt{\hbar}} \cos \left( \frac{1}{\hbar} \int_{a}^{x} p dx - \frac{\pi}{4} \right)$. The semiclassical approach is valid to the both sides of the point $x = a$. It is also valid at small right vicinity of the point $x = 0$ provided that the potential behaves good enough at that region. To the left side of $x = 0$ one must holds $\Psi_{x < 0}(x) = 0$. Using the continuity argument, we have $\Psi_{x > 0}(x) = \frac{C'}{\sqrt{\hbar}} \cos \left( \frac{1}{\hbar} \int_{0}^{x} p dx - \frac{\pi}{2} \right)$. Now remember that $\Psi_{x > 0}(x) \equiv \Psi_{x < a}(x)$, and, proceeding in the same way as for the usual Bohr-Sommerfeld rule, we obtain,

$$\frac{1}{\hbar} \left( \int_{0}^{a} + \int_{a}^{\infty} \right) p dx - \frac{3\pi}{4} = \pi n,$$

(1)

or

$$\oint p dx = 2\pi\hbar \left( n + \frac{3}{4} \right).$$

(2)

![Fig. 1 – Potentials](image)

2. Using the quantization condition deduced in the previous point, we have,

$$\int_{0}^{a} \sqrt{2m(E - V_{0} - kx)} dx = \frac{\sqrt{2m}}{k} (E - V_{0})^{3/2} \int_{0}^{a} \sqrt{1 - y dy} =$$

$$\frac{2}{3} \frac{\sqrt{2m}}{k} (E - V_{0})^{3/2} = \pi \hbar \left( n + \frac{3}{4} \right),$$

(3)
and
\[ E_n = V_0 + \left( \frac{3k}{2 \sqrt{2m}} \pi \hbar \left( n + \frac{3}{4} \right) \right)^{2/3}, \]

where \( m = m_e/2 \) is a reduced mass of the two-quark system. Numerically,
\[ E_n \approx 1.42 \cdot n^{2/3} \text{ GeV}. \]

(4)

(5)

2. Barriers

Recall the definition of the transmission coefficient, 
\[ D(E) = \exp \left( i \frac{\pi}{2} \right) \exp \left( -\frac{1}{\hbar} \int_a^b |p| dx \right), \]
where \( a, b \) are matching points.

1. For the potential on Fig.2a we have,
\[
\int_a^b \sqrt{2m} \left( V_0 \left( 1 - \frac{x^2}{x_0^2} \right) - E \right) dx = x_0 \sqrt{2mV_0} \int_a^b \sqrt{1 - \frac{E}{V_0} - \frac{x^2}{x_0^2}} dx = \]
\[
x_0 \sqrt{2mV_0} \left\{ \frac{x}{x_0} \sqrt{1 - \frac{E}{V_0} - \frac{x^2}{x_0^2}} + \left( 1 - \frac{E}{V_0} \right) \tan^{-1} \left( \frac{x}{x_0} \sqrt{1 - \frac{E}{V_0} - \frac{x^2}{x_0^2}} \right) \right\} \bigg|_{x=a}^{x=b} = \]
\[
x_0 \sqrt{2mV_0} \frac{1}{2} \pi \left( 1 - \frac{E}{V_0} \right), \]

where we used the relations \( 1 - a^2/x_0^2 = E/V_0 \) and \( 1 - b^2/x_0^2 = E/V_0 \). Finally,
\[ D(E) = \exp \left( i \frac{\pi}{2} \right) \exp \left( -\frac{x_0}{2\hbar} (1 - E/V_0) \pi \sqrt{2mV_0} \right). \]

(6)

(7)

2. For the potential on Fig.2b the calculation is similar:
\[
\int_0^\infty \sqrt{2m}(-Fx - E) dx = \frac{2}{3} \sqrt{2m} \frac{(-E^{3/2})}{F}, \]

and
\[ D(E) = \exp \left( i \frac{\pi}{2} \right) \exp \left( -\frac{2}{3\hbar} \sqrt{2m} \frac{(-E)^{3/2}}{F} \right). \]

(8)

(9)
3. Cubic potential

The life time of the particle in the well is determined by the tunneling probability, \(|D(E)|^2\). The energy \(E\) is assumed to be small compared with the height of the barrier, \(E \ll V_{\text{max}} = \frac{4}{27} V_0 x_1^3\). With this assumption, the integration in \(D(E)\) is performed easily,

\[
\int_a^b \sqrt{2m \left(V_0(x^2 x_1 - x^3) - E\right)} dx = \sqrt{2m V_0} \int_a^b \sqrt{x^2 x_1 - x^3 - \frac{E}{V_0}} dx \approx \sqrt{2m V_0} \left[ \sqrt{x_1 - x(3x^2 - x_1 x - 2x_1^2)} \right]_0^{x_1} = \frac{4}{15} \sqrt{2m V_0 x_1^{5/2}}. \tag{10}
\]

In the second line we have neglected the term \(E/V_0\), since its contribution to the integral becomes essential only near the turning points \(a\) and \(b\), where the integrand is small. For the same reason, we have approximated the limits of integration as \(a \approx 0, b \approx x_1\).

The life time is given by \(\tau = T/|D(E)|^2\), where \(T\) is the period of classical oscillations in the well,

\[
T = \oint dt = 2 \int_{a_1}^{a_2} \frac{mdx}{p(x)} \approx \sqrt{\frac{2m}{E}} \int_{a_1}^{a_2} \frac{dx}{\sqrt{1 - \frac{x_1 V_0}{E} x^2}} = \sqrt{\frac{2m}{x_1 V_0} \left[ \arcsin \sqrt{\frac{x_1 V_0}{E} x} \right]_{a_1}^{a_2}} = \sqrt{\frac{2m}{x_1 V_0} \pi}. \tag{11}
\]

Again, since \(E \ll V_0\), one can use parabolic approximation for the potential near the origin: \(V \approx V_0 x_1 x^2, x \approx 0\), and the turning points \(a_1, a_2\) are determined from the equation \(E = V_0 x_1 a_{1,2}^2\) and equal \(a_1 = -a_2 = \sqrt{\frac{E}{V_0 x_1}}\). Finally,

\[
\tau = \sqrt{\frac{2m}{x_1 V_0} \pi} \exp \left( \frac{8}{15\hbar} \sqrt{2m V_0 x_1^{5/2}} \right). \tag{12}
\]

4. Ground state

Given below is a "handwaving" proof of the statement. Despite the lack of mathematical formulas, it is still quite rigorous.

If \(V(x)\) contains bound states, it must have a minimum at finite \(x, V_0 = V(x_0)\), see Fig.1b. In a small neighbourhood of \(x_0\), \(V(x)\) can be approximated by parabolic function (or even constant). Consider two infinite walls, standing close to each other, at points \(x_1\) and \(x_2\), with the point \(x_0\) between them. If the distance \(\Delta = x_2 - x_1\) is small enough, so that \(V(x) \approx \text{const}\) for \(x_1 < x < x_2\), we have a "box", for which the wave functions are known very well, \(\Psi_n(x) \sim \sin \left( \frac{2\pi}{a_1} (x - x_1) \right), x_1 < x < x_2\), and \(\Psi_n(x) = 0\) otherwise. The ground state wave function in this box, \(\Psi_1(x)\), obviously, has no zeros between \(x_1\) and \(x_2\).
Now we start to move the walls apart. As \( \Delta \) grows, \( V(x) \) is no longer constant between the points \( x_1 \) and \( x_2 \), and the bound states are no longer equal to those for the rectangular box. While \( \Delta \) increases, \( \Psi_n(x) \) resemble the true bound state wave functions of the potential \( V(x) \). We should prove that during the process of "separation" of the walls the ground state wave function \( \Psi_1(x) \) cannot acquire any nodes. In the limit \( \Delta \to \infty \) it approaches the ground state wave function of \( V(x) \) which, hence, has no zeros (at any finite region of \( x \)).

Assume, on the contrary, that it does. There are two possibilities for this: either at least one of the derivatives at \( x_1, x_2 \) must change sign; or, the derivatives at \( x_1, x_2 \) do not change sign, but the wave function itself develops two zeros through its deepening at some point between \( x_1 \) and \( x_2 \). In both cases there will be a critical value of \( \Delta \) such that the wave function and its first derivative vanish at the same point. In the first case, this is one of the points \( x_1, x_2 \), and in the second case, there must be a value of \( \Delta \) such that the wave function touches the X-axis, just before it dips down and develops two zeros. But, since the Schrödinger equation is a linear, second-order, ordinary differential equation, it has a unique solution, given the value of the function and its derivative at the same point. But if the solution and its first derivative are zero at the same point, one concludes that the wave function must be identically zero. As far as we know, that there is always a non-trivial solution of the Schrödinger equation in one-dimensional potential well, we conclude that in both cases ground state function can develop no nodes.

The same argument shows that, since \( n \)-th wave function starts with \( n - 1 \) zeros, their number can neither increase nor decrease, since in order to develop a new zero, or "lose" a zero, the wave function must go through one of the stages described above.