1. Completeness of Coherent states

We know from the lecture notes that the eigenstates $|n\rangle$ of the harmonic oscillator form a complete basis in the Hilbert space $\mathcal{H}$. This means that every state from $\mathcal{H}$ can be represented by a linear combination of some of $|n\rangle$. The completeness can also be expressed as a resolution of identity:

$$\sum_n |n\rangle\langle n| = 1. \quad (1)$$

Is the set of coherent states $|\alpha\rangle$, $\alpha \in \mathbb{C}$, also complete? To answer this question, one should prove the relation similar to (1) for them.

1. Show first that

$$\int d^2\alpha e^{-|\alpha|^2} \alpha^n \alpha^m = \pi n! \delta_{nm}, \quad (2)$$

where the integration measure $d^2\alpha$ is defined as

$$d^2\alpha \equiv d\text{Re}\alpha d\text{Im}\alpha. \quad (3)$$

*Hint*: Use polar coordinates.

2. Now prove that

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle \alpha| = 1. \quad (4)$$

*Note*: Hence we find that the set of coherent states is complete. In fact, however, it is overcomplete in the sense that the decomposition of some state from $\mathcal{H}$ into the linear combination of $|\alpha\rangle$ is not unique. Hence not all $|\alpha\rangle$ are linearly independent.

3. To demonstrate explicitly the last statement, find the real function $v(|\alpha|)$ such that

$$\int d^2\alpha v(|\alpha|)|\alpha\rangle = |0\rangle. \quad (5)$$

*Note*: It turns out that to form a complete basis, it is enough to take the subset of the set of coherent states of the form $|n + im\rangle$, $n, m \in \mathbb{Z}$ with one and only one $|n + im\rangle$ excluded.

4. * To see this, find $c_{nm} \in \mathbb{R}$ such that

$$\sum_{n,m} c_{nm} |n + im\rangle = |0\rangle. \quad (6)$$
2. The saddle–point method

In quantum physics one often finds the integral \( I = \int_{-\infty}^{\infty} dx \ e^{i\hbar f(x) - \epsilon|x|} \), where \( f(x) \) is some function with “good” behavior at \( x \in (-\infty, \infty) \) (i.e. it and its derivatives are continuous with their absolute values bounded from above, and tend to zero as \( x \) approaches \( \pm \infty \)), and \( \epsilon \) is small positive number added to ensure the convergence of the integral. When \( \hbar \to 0 \), the integrand becomes a rapidly oscillating function. In this limit, the integral \( I \) can be evaluated at the extremum point of \( f(x) \). This procedure of computing the integrals is called the saddle–point method, or the stationary phase method or the steepest–descent method.

1. Calculate \( I \) up to order \( O(\hbar^{1/2}) \).
2. In some cases, the saddle–point method gives an exact result. Show that this is indeed the case when the integrand is a Gaussian function.

3. Classical actions

One of key ingredients of Path integral approach is the action calculated on some classical solution of the equations of motion. Consider a classical trajectory \( x(t) \) of a particle with boundary conditions \( x(0) = x_i, x(T) = x_f \), which obeys the classical equations of motion. Denote \( S[x(t)] \equiv S_{cl}(x_i, 0; x_f, T) \).

1. Calculate \( S_{cl}(x_i, 0; x_f, T) \) for free particle, \( L = \frac{m\dot{x}^2}{2} \).
2. Calculate \( S_{cl}(x_i, 0; x_f, T) \) for harmonic oscillator, \( L = \frac{m}{2}(\dot{x}^2 - \omega^2x^2) \).