The assistant for this course, Jan Cristina, does not speak fluent French. He can understand and communicate in basic French however.

We introduce the following topological notation. The frontier of a set in a metric space is defined as \( \text{Fr}(U) := \overline{U} \setminus \text{Int}(U) \). Usually in metric and point set topology this is called the boundary, and denoted \( \partial(U) \), but to distinguish it from the manifold boundary we denote it \( \text{Fr}(U) \). In general for these exercises, we assume the metric space definition of topology. Unfortunately this means we will miss some of the subtleties and interesting counterexamples of spaces which are like manifolds, but missing one crucial component such as the long line.

Recall that a space \( X \) is connected if it cannot be expressed as the union of two disjoint open subsets. A connected component of a space \( X \) is a maximal connected set. If an open set is connected then it is also closed. The connected component of a space \( X \) is an open and closed subset \( U \) which does not strictly contain any open and closed subset except \( \emptyset \).

We recall the following notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
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<tr>
<td>( \mathbb{R}^n )</td>
<td>Euclidean space of dimension ( n )</td>
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<tr>
<td>( \mathbb{H}^n )</td>
<td>( {(x_1,\cdots,x_n) \in \mathbb{R}^n \text{ s.t. } x_1 &gt; 0} )</td>
</tr>
<tr>
<td>( \mathbb{H}^n )</td>
<td>( {(x_1,\cdots,x_n) \in \mathbb{R}^n \text{ s.t. } x_1 \geq 0} )</td>
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<tr>
<td>( \mathbb{B}^n )</td>
<td>( {x \in \mathbb{R}^n \text{ s.t. }</td>
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Lastly differential topology is a subject which lends itself to reduction to the “obvious”, such as the fact that \( \mathbb{R}^n \) and \( \mathbb{R}^m \) are homeomorphic if and only if \( m = n \), however the proof of invariance of domain is anything but obvious. If you are tempted to use the phrases “obviously” or “clearly”, recall the definition of obvious:

A statement is “obvious” if a proof springs immediately to mind.

Timothy Gowers. (paraphrased)

**Exercice 1.1.** Show that the number of connected components of a metric space is invariant under homeomorphism. Without using invariance of domain, show that \( \mathbb{R} \) is not homeomorphic to \( \mathbb{R}^n \) for any \( n \).

**Solution 1.1.** Let \( f : X \to Y \) be a homeomorphism between metric spaces. Let \( U \) be a connected component of \( X \), then \( f(U) \) is a connected component, otherwise any decomposition of \( f(U) \) can be pulled back by \( f^{-1} \). Similarly we can start from a connected component of \( Y \). Thus \( U \mapsto f(U) \) as a set map forms a bijection of the connected components.

Now suppose there is a homeomorphism \( f : \mathbb{R} \to \mathbb{R}^n \) for some \( n \geq 2 \). Then \( f|_{\mathbb{R}\setminus\{0\}} \) is a homeomorphism from \( \mathbb{R}\setminus\{0\} \to \mathbb{R}^n \setminus \{f(0)\} \). But \( \mathbb{R}\setminus\{0\} \) has two components, while \( \mathbb{R}^n \setminus \{x\} \) has only one component for \( n \geq 2 \).

**Exercice 1.2.** For some \( n \) give an example of a closed subset of \( \mathbb{R}^n \) which is not a manifold. Prove that it is not a manifold.
Solution 1.2. Any solution will do, but a relatively easy example is \(X = \{(x, y) \in \mathbb{R}^2 \mid x = 0 \text{ or } y = 0\}\). To prove it is not a manifold consider some neighbourhood \(U\) of \((0, 0)\). Suppose there is a homeomorphism \(f\) of this neighbourhood to some \(\mathbb{R}^n\). By assumption there is an open ball \(B\) around \(f((0, 0))\) contained in \(f(U)\). Then \(B \setminus \{f((0, 0))\}\) has at most 2 components, while \(f^{-1}(B) \setminus \{(0, 0)\}\) has at least four.

Another good example is the Cantor set. This set has many constructions but the most classical is to consider the set \(T_{n+1} = T_n/3 \cup (T_n/3 + 1/3)\), where \(T_0 = [0, 1] \subset \mathbb{R}\). Let \(C = \bigcap_{n=0}^{\infty} T_n\). This is an example of a totally disconnected set. Every point has a sequence of separate points which converges to it, and every point is a connected component. If any point had a neighbourhood homeomorphic to some \(\mathbb{R}^n\) then there would be a homeomorphism from some ball to a neighbourhood of that point. But the neighbourhood of any point has an infinite number of connected components.

Exercice 1.3. For some \(n\), give an example of an open subset of \(\mathbb{R}^n\), and hence a manifold, whose frontier is not a manifold.

Solution 1.3. We can used the previous example and consider the open set \(U = \{(x, y) \in \mathbb{R}^2 \mid x \cdot y > 0\}\). Its boundary is given by the space \(X\) in the preceding example. Alternatively one can use the open set \([0, 1] \setminus C\).

Exercice 1.4. Consider the sphere \(S^{n-1}\)

\[S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}\]

In lectures this was shown to be a manifold. What is the minimum number of charts in an atlas?

Solution 1.4. We construct an atlas with two charts. Consider the map

\[\varphi_1 : B(0, 2\pi/3) \subset \mathbb{R}^{n-1} \to \mathbb{R}^n, \quad x \mapsto (\frac{x}{|x|} \sin |x|, \cos |x|)\]

This is continuous because it is a product and sum of continuous functions (\(\sin |x|/|x|\) is continuous by l’hôpital’s rule. Note that \(\sin\) is non-negative on \([0, 2\pi/3]\). Hence

\[x_1 = x_2 \quad \frac{\sin |x_1|}{|x_1|} = x_2 \quad \frac{\sin |x_1|}{|x_2|}\]

is the case only if \(x_1/|x_1| = x_2/|x_2|\) and \(\sin |x_1| = \sin |x_2|\) or \(|x_1| = 0 = |x_2|\). But if \(x_1 \neq x_2\) then the latter can hold only when \(\cos |x_1| = -\cos |x_2|\). The set \(\overline{B}(0, 2\pi/3)\) is compact and \(\varphi_1\) is injective so the map is a homeomorphism onto its image, covering \(S^{n-1} \cap \{(x, y, z) \mid z \geq -0.5\}\). If we restrict to \(B(0, 2\pi/3)\) the map is still a homeomorphism, as is its inverse.

The other chart is given by the inverse of

\[\varphi_2 : B(0, 2\pi/3) \subset \mathbb{R}^{n-1} \to \mathbb{R}^n, \quad x \mapsto (\frac{\sin |x|}{|x|}, -\cos |x|)\]

on its image. The argument by which it is a homeomorphism is exactly duplicated.

To see that there can be no atlas with one chart, one need only note that \(S^2\) is a compact space, and so the image of any homeomorphism must be compact, but there are no compact open sets on in \(\mathbb{R}^2\).

Exercice 1.5. Let \(K\) be a bounded open convex set in \(\mathbb{R}^n\) by which we mean for any \(x\) and \(y\) in \(K\), and any \(t \in [0, 1]\)

\[t \cdot x + (1 - t) \cdot y \in K\]

Show that the \(\text{Fr}(K)\) is a manifold.
Solution 1.5. As per the hint, we assume that $0 \in K$. By openness there is an $r > 0$ such that $B(0, r) \subset K$. By choosing $r$ a little bit smaller, we may assume $B(0, r) \subset K$. By boundedness we know there is an $R$ such that $K \subset B(0, R)$. We introduce a function

$$|x|_K := (\sup \{\lambda : \lambda x \in K\})^{-1}.$$  

Then we consider the function $x \mapsto x/|x|_K$ for $x \in S^{n-1} \subset \mathbb{R}^n$. We claim first that $r|x| \leq |x|_K \leq R|x|$. To see this note that $xr/|x| \in \overline{B}(0, r) \subset K$, and $xR/|x| \in \mathbb{R}^n \setminus B(0, R)$ so $xR/|x| \notin K$.

We claim that $x/|x|_K \in \partial K$. Suppose $x/|x|_K \in \text{Int}(K)$ then there is an epsilon such that $x/(|x|_K - \epsilon) \in \int(K)$, but this violates the definition of $|x|_K$. Now suppose that $x/|x|_K$ is in the complement of $K$. Then there is an $\epsilon$ such that for all $x/(|x|_K + \epsilon)$ is in the complement of $K$, again contradicting the construction.

To show that $x/|x|_K$ is continuous we must show that $x \mapsto |x|_K$ is continuous and non zero on $S^{n-1}$, but we showed that

$$|x|_K \geq r|x| \geq r > 0$$

if $x \in S^{n-1}$.

To show that $\cdot \cdot |K$ is continuous we show first that it satisfies

$$|x + y|_K \leq |x|_K + |y|_K.$$

To see this, let $t = |x|_K/(|x|_K + |y|_K)$ so $(1-t) = |y|_K/(|x|_K + |y|_K)$. Then

$$tx + (1-t)y = \frac{x + y}{|x|_K + |y|_K}$$

is in $K$. Consequently

$$1/(|x|_K + |y|_K) \in \{\lambda : \lambda (x+y) \in K\},$$

and so

$$1/\lambda + |y|_K \geq 1/(|x|_K + |y|_K),$$

which implies

$$|x + y|_K \leq |x|_K + |y|_K.$$

Consider $||x|_K - |y|_K| = \max\{|x|_K - |y|_K, |y|_K - |x|_K\}$. Now let $z = x - y$

$$|x|_K - |y|_K = |z + y|_K - |y|_K \leq |z|_K \leq R|x - y|$$

and

$$|y|_K - |x|_K = | - z + x|_K - |x|_K \leq | - z|_K \leq R|x - y|.$$  

Thus for $\delta = \epsilon/R$, any $x, y$ such that $|x - y| \leq \delta$ implies $||x|_K - |y|_K| \leq \epsilon$, so $|x|_K$ is continuous, and the map $x \mapsto |x|_K$ is continuous. But we can restrict the map $x \mapsto x/|x|$ (the usual norm) to $\partial K$ to get an inverse map, which by similar arguments is continuous. i.e.

$$(x/|x|_K)/(|x|/|x|_K) = x/|x| = x,$$

if $x \in S^{n-1}$. Consequently $\partial K$ is homeomorphic to $S^{n-1}$, a manifold.

Exercice 1.6. Let $K$ be an open convex subset of $\mathbb{R}^n$ show that $K$ is homeomorphic to $\mathbb{R}^n$.  

Solution 1.6. Again assume that $0 \in K$ (if not apply the homeomorphism of translation). Now for every $x \in K$ consider the map $\psi(x) = \inf\{|x|\lambda - 1|^{-1} \mid \lambda x \in K, \lambda > 1\}$. Then let $f : x \mapsto x\psi(x)$. Now we will show that $f$ is continuous. Set $f(0) = 0$. By openness there is a ball $B(0, r) \subset K$. So for any $x \in B(0, er) \mid f(x)\mid \leq |x|\epsilon/(1 - \epsilon) \leq \epsilon$ for $\epsilon$ sufficiently small. Hence $f$ is continuous at 0. To see that $f$ is continuous at $x \neq 0$ consider that there is some $R$ such that $B(x, 2R) \subset K$. Then for $y \in B(x, R/2)$ it follows that $B(y, R/2) \subset K$. Let $\sigma = \max\{|x|, |y|\}$, then for and for $\lambda < 1 + R/(\sigma) \lambda y \in K$, and hence $\psi(y) \leq (\sigma + R)/R = C$.

Let $y \in B(x, R/2)$. Now for any $\lambda < \psi(x)/(\psi(x) - 1) \lambda x \in K$. And for any $\mu < \psi(y)/(\psi(y) - 1) \leq 1 + R/\sigma \mu y \in K$. Now consider the line from $\mu y$ to $\lambda x$ :

$$\lambda x + t(\mu y - \lambda x).$$

Consequently consider $z \in B(x, R\delta/2)$, let $y = x + (z - x)/\delta$, then $y \in B(x, R/2)$, and so for any $t$

$$\lambda x + t(\mu(x + (z - x)/\delta) - \lambda(x))$$

$$(1 - t)\lambda + t\mu(1 - 1/\delta)x + t\mu z/\delta$$

is in $K$. Assume $t = \lambda/((\delta - 1)\mu + \lambda)$. It follows that for $\nu < \mu\lambda/(\delta\lambda + \mu(1 - \delta)) \nu z \in K$. Let $\mu\delta = (1 - \delta)\mu$, so

$$\psi(z) \leq \frac{\mu\lambda}{\mu\lambda - \delta\lambda - \mu\delta} \leq \frac{\lambda}{\lambda - 1 - \frac{1}{\mu\lambda(\lambda - 1)}}$$

$$\leq \frac{\lambda}{\lambda - 1}(1 + 2\delta\lambda/(\mu\delta(\lambda - 1)))$$

for $\delta\lambda/(\mu\delta(\lambda - 1)) < 1/2$.

Note we use the approximation

$$1/(1 - x) = \sum_{i=0}^{\infty} x^i \leq 1 + 2x,$$

for $x \leq 1/2$. Hence letting $\lambda$ approach $\psi(x)/(\psi(x) - 1)$, yields $\psi(z) \leq \psi(x)(1 + 2\delta\psi(x))$ provided $\delta < \mu/2\psi(x)$. Choose $\delta < \mu\epsilon/\psi(x)^2$ and let $\mu$ tend to $1 + R/\sigma$ implies the result one way, but similarly we can bound $\psi(x) \leq \psi(z)(1 + 2\delta\psi(z)/$, and by choosing $\delta < \epsilon\mu/C^2$ implise for $\psi(x) - \psi(z) \leq \epsilon$, which yields continuity of $\psi$ at $x$ for $x \neq 0$.

Lastly we show that $\psi$ is bijective. For $\tau \in S^{n-1}$ consider the set $\ell_\tau := \{t\tau \mid t \in [0, \infty]\}$. Then $f$ induces a bijection from $\ell_\tau \cap K$ to $\ell_\tau$. Let us consider two cases either $\ell_\tau \subset K$, in which case $\psi(x) = 1$ for all $x \in \ell_\tau$, or $\lambda = \sup\{|x| \mid x \in \ell_\tau \cap K\} < \infty$. In the first case $f$ is the identity. In the second case $|f(t\tau)| \mapsto \lambda t/|\lambda - t|$ maps $[0, \lambda]$ bijectively to $[0, \infty]$, (it has an inverse $s \mapsto \lambda s/(s - 1)$). Because the directions cannot be positive multiples of one another, $f$ is a bijection. Now for each $x$ in the $\mathbb{R}^n$ there is a $y$ which maps to $x$ under $f$. By invariance of domain the image of a continuous injective map $\mathbb{R}^n \to \mathbb{R}^n$ is open and in particular is a homeomorphism.

Exercice 1.7. Consider the following subset $T^2 \subset \mathbb{R}^3$ given by

$$T^2 = \left\{ \begin{pmatrix} \cos(t)(2 + \cos(s)) \\ \sin(t)(2 + \cos(s)) \\ \sin(s) \end{pmatrix} \mid s \in [0, 2\pi], t \in [0, 2\pi] \right\}.$$

Show that it is a manifold. What is the minimum number of charts in an atlas?

Solution 1.7. Consider the set $U = B(0, 5\pi/3) \setminus B(0, \pi/3)$. Consider the maps

$$\psi^+: x \mapsto (|2 + \sin(|x| - \pi)|x/|x|, \cos(|x| - \pi)),$$
\[ \psi^- : x \mapsto (2 + \sin(|x| - \pi))x/|x|, -\cos(|x| - \pi)), \]

Firstly all of the functions are continuous on the domain of definition because \(|x| > \pi/3 > 0\). So we must only show surjectivity and injectivity. Injectivity follows from

\[ 2 + \sin(q) \geq 1 > 0. \]

Hence if \( \psi^+(x_1) = \psi^+(x_2) \) then \((2 + \sin(|x_1| - \pi)) = (2 + \sin(|x_2| - \pi)) \) and \( x_1/|x_1| = x_2/|x_2| \). But for the domain of these functions \( \sin(|x_1| - \pi) = \sin(|x_2| - \pi) \) and \( \cos(|x_1| - \pi) = \cos(|x_2| - \pi) \) if and only if \(|x_1| = |x_2| \). Now for any \( t \in [0, 2\pi] \) and \( s \in [0, \pi] \), choose the chart \( \psi^+ \). Let \(|x| = \pi/2 + s \), and let \( x/|x| = (\cos t, \sin t) \). Then

\[ \psi^+(x) = \begin{pmatrix} \cos(t)(2 + \cos(s)) \\ \sin(s) \end{pmatrix}. \]

If \( s \in \pi, 2\pi \) then choose the chart \( \psi^- \), and choose \(|x| = s - \pi/2 \).

Once again both maps can be defined on \( U \subset \mathbb{R}^2 \). They are still injective and continuous, and hence homeomorphisms onto their image. Restricting to \( U \) again then the maps are still homeomorphisms.

Similarly to the sphere case \( T^2 \) is compact, because it is the continuous image of a compact set. Hence it cannot be homeomorphic to an open subset of \( \mathbb{R}^2 \).

**Exercice 1.8.** Let \( M \) be a compact topological \( n \)-manifold with boundary. Show that \( \partial M \) is a compact topological \( n - 1 \) manifold, and that \( \partial(\partial M) = \emptyset \).

**Solution 1.8.** It is evident that \( \partial M \) is a Hausdorff topological space with a countable base of open sets. For every point \( p \in \partial M \), there exists a local chart \((\phi, U)\) such that \( \phi(p) \in \mathbb{R}^{n-1} \times \{0\} \). Moreover, by the invariance of the boundary, \( \phi : \partial M \cap U \to \mathbb{R}^{n-1} \times \{0\} \). Thus \((\phi, U \cap \partial M)\) is a local chart for \( \partial M \) which takes values in \( \mathbb{R}^{n-1} \), and not \( \mathbb{H}^{n-1} \); hence there is no point mapped to \( \partial \mathbb{H}^n \) and \( \partial(\partial M) \) is empty.

**Exercice 1.9.** Prove that if \( M \) and \( N \) are topological manifolds with boundaries, then \( M \times N \) is also a topological manifold with boundary. Show that

\[ \dim(M \times N) = \dim(M) + \dim(N) \quad \partial(M \times N) = \partial M \times N \cup M \times \partial N. \]

**Solution 1.9.** The topological space \( M \times N \) is evidently Hausdorff and has a countable basis of open sets. Take \( (x, y) \in M \times N \), and suppose that \( x \notin \partial M \) and \( y \notin \partial N \). Then there exist local charts \( \phi_M : U_1 \to \mathbb{R}^m \) and \( \phi_N : U_2 \to \mathbb{R}^n \) around the points \( x, y \). Consider the map defined on the open set \( U_1 \times U_2 \) into \( \mathbb{R}^{m+n} \) defined by

\[ \phi(s, t) \equiv (\phi_M(s), \phi_N(t)). \]

This map is evidently injective, and by definition of product topology it is also a local homeomorphism.

Now suppose that \( x \in \partial M \) and \( y \notin \partial N \). Then with the same construction we find a local chart on \( \mathbb{H}^{m+n} \), and in a similar way we can deal with the case \( x \notin \partial M \) and \( y \in \partial N \).

If both \( x \) and \( y \) belong to the boundary of their respective manifolds, then the construction just described provides a local homeomorphism of \( U_1 \times U_2 \) into an open subset of

\[ \{(s_1, \ldots, s_n, t_1, \ldots, t_m) \in \mathbb{R}^{n+m} \text{ s.t. } s_1 \geq 0, t_1 \geq 0\}. \]

This is not exactly a half space, but it is easy to construct a homeomorphism between this set and a half space.
Consider the homeomorphism which leaves the \( m + m - 2 \) coordinates \( s_2, \ldots, s_n, t_2, \ldots t_m \) fixed, and transforms the other two by

\[
\begin{align*}
\text{if } s_1 &\geq t_1 \\
\begin{cases}
s'_1 &= s_1 - \frac{s_1 - t_1}{\sqrt{4}} \\
t'_1 &= t_1 - \frac{s_1 - t_1}{\sqrt{4}}
\end{cases} \\
\text{if } s_1 &< t_1 \\
\begin{cases}
s'_1 &= s_1 - \frac{t_1 - s_1}{\sqrt{4}} \\
t'_1 &= t_1 - \frac{t_1 - s_1}{\sqrt{4}}
\end{cases}
\end{align*}
\]

This maps onto the set \( s'_1 + t'_1 \geq 0 \) but by rotating we arrive at our appropriate homeomorphism.

**Exercice 1.10.** Show that if a manifold \( M \) is connected, then it is path connected, i.e. every two points \( x \) and \( y \) in \( M \), can be connected by a continuous map \( \gamma : [0, 1] \to M \) such that \( \gamma(0) = x \) and \( \gamma(1) = y \).

**Solution 1.10.** To prove this, consider the set of points which can be connected to some \( x \in M \) by a path. Call this set \( U_x \). To see that this set is open, pick a point \( z \in U_x \). There is a chart \( \varphi : U \to \mathbb{R}^n \) around \( z \) which maps to a ball. Thus any point \( z' \in U \) can be connected to \( x \) via a path connecting to \( x \) to \( z \) and one connecting \( z \) to \( z' \). If \( M \) is connected then the complement of \( U_x \) in \( M \) can either be \( \emptyset \), or it cannot be open. If the complement is empty we are done. If not by the assumption that \( M \) is connected, \( U_x \) cannot be closed, so the \( \overline{U_x} \) contains a point not in \( U_x \), i.e. there is an element \( y \in \text{Fr}(U_x) \). But then we can find a path connected neighbourhood \( V \) of \( y \). But this neighbourhood has to intersect \( U_x \) otherwise \( U \setminus V \) is a closed set containing \( U \) which is a contradiction. Pick \( z \in U \cap V \).

Then \( z \) can be connected to \( y \) because \( V \) is path connected, and \( z \) can be connected to \( z' \) because \( U \) is path connected. Contradicting our assumption the complement is non-empty.

**Hints.** Remember that to show that something is a manifold, you can show this it has an atlas of charts, or you can show it is homeomorphic to a known manifold. Also when constructing charts it is worth remembering that not all open sets are balls. For open convex sets, remember that you can always fit a ball in them, and assume without loss of generality that the ball is centered in the origin. What can you say about the intersection of rays from the origin and the convex set?

**Addenda.** Exercise 1.4, the balls \( B(0, 2\pi/3) \) have been specified to be in \( \mathbb{R}^{n-1} \), to better be charts for an \( n - 1 \)-dimensional manifold. Exercise 1.5 had the notation \( \int(K) \) this has been replaced with the intended \( \text{Int}(K) \) for the interior of \( K \). Thank you to the students who pointed out these errors. Exercise 1.6 had the estimate \( |f(x)| < |x|/r \) has been replaced with \( |f(x)| < |x|\epsilon/(\epsilon - 1) \).