Problem 1.

i) C, because $\mathbb{R}$ is uncountable.

ii) C, because $\mathbb{R}$ is uncountable.

iii) B, because $\mathbb{Q}$ and $\mathbb{Z}$ are infinite and countable.

iv) A, because this is the empty set.

v) A, because the only even prime number is 2.

Problem 2.

i) B, since

$$F(x^2) = \sum_{i=0}^{\infty} f_i(x^2)^i = \sum_{i=0}^{\infty} f_i x^{2i} = f_0 + 0 \cdot x + f_1 x^2 + 0 \cdot x^3 + \ldots$$

ii) M, since

$$xF(x) = x \sum_{i=0}^{\infty} f_i x^i = \sum_{i=0}^{\infty} f_i x^{i+1} = 0 + f_0 x + f_1 x^2 + f_2 x^3 + \ldots$$

iii) O, since

$$F'(x) = \sum_{i=1}^{\infty} i f_i x^{i-1} = f_1 + 2 f_2 x + 3 f_3 x^3 + \ldots$$

iv) J, since

$$\frac{F(x) + F(-x)}{2} = \frac{1}{2} \left( \sum_{i=0}^{\infty} f_i x^i + \sum_{i=0}^{\infty} f_i (-x)^i \right)$$

$$= \frac{1}{2} \left( 2 f_0 + (f_1 - f_1) x + 2 f_2 x^2 + (f_3 - f_3) x^3 + \ldots \right) = f_0 + 0 \cdot x + f_2 x^2 + 0 \cdot x^3 + \ldots$$

Problem 3.

i) C, since

$$11^7 \equiv (-2)^7 \equiv (-2) \cdot 4^3 \equiv (-2) \cdot 3 \cdot 4 \equiv (-2) \cdot (-1) \equiv 2 \mod 13.$$
ii) Since
\[ 3^{170} \equiv 3 \cdot (3^{13})^{13} \equiv 3 \cdot 3^{13} \equiv 3 \cdot 3 \equiv 9 \mod 13. \]

Problem 4.
We make 11 boxes, for eating 0, 1, 2, . . . , 10 worms, respectively. The smart pigeons have to be distributed in the boxes 6 to 10, the not-so-smart pigeons in the boxes 0 to 4, i.e., box 5 remains empty. We have \( k \) smart pigeons and \( n - k \) not-so-smart pigeons, and 5 boxes each. By the pigeonhole principle, we know that at least \( \lceil k/5 \rceil \) smart pigeons are in the same box, and at least \( \lceil (n - k)/5 \rceil \) not-so-smart pigeons are in the same box. Since \( k < n/2 \), we obtain \( \lceil (n - k)/5 \rceil \geq \lceil k/5 \rceil \), thus the best lower bound is \( \lceil (n - k)/5 \rceil \). Therefore, answer C is correct.

Problem 5.

1. T
2. F
3. F
4. F
5. F
6. F
7. F
8. T

Problem 6.
Let \( A \) be the set of integers that are divisible by 6. Then
\[ |A| = \left\lfloor \frac{2015}{6} \right\rfloor = 335. \]

Let \( B \) be the set of integers that are divisible by 7. Then
\[ |B| = \left\lfloor \frac{2015}{7} \right\rfloor = 287. \]

Let \( C \) be the set of integers that are divisible by 8. Then
\[ |C| = \left\lfloor \frac{2015}{8} \right\rfloor = 251. \]

Moreover, the set of integers that are divisible by 6 and 7 has cardinality
\[ |A \cap B| = \left\lfloor \frac{2015}{42} \right\rfloor = 47, \]
the set of integers that are divisible by 6 and 8 has cardinality
\[ |A \cap C| = \left\lfloor \frac{2015}{24} \right\rfloor = 83, \]
the set of integers that are divisible by 7 and 8 has cardinality
\[ |B \cap C| = \left\lfloor \frac{2015}{56} \right\rfloor = 35. \]
Furthermore, the set of integers that are divisible by 6, 7 and 8 has cardinality
\[ |A \cap B \cap C| = \left\lfloor \frac{2015}{168} \right\rfloor = 11. \]

Inclusion-exclusion principle implies that the number of integers that are divisible by 6, 7, or 8 is
\[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \]
\[ = 335 + 287 + 251 - 47 - 83 - 35 + 11 = 719. \]

**Problem 7.**
We prove this by induction.
**Induction Basis:** For \( n = 0 \) we obtain
\[ h_0(x) = x(1 - x) = (1 - x)x = h_0(1 - x). \]
**Inductive Step** \( n \to n + 1 \): We assume as induction hypothesis that \( h_n(x) = h_n(1-x) \).
Then we obtain
\[ h_{n+1}(x) = h_n(x^2) + h_n(x(2-x)) \]
\[ \overset{IH}{=} h_n(1-x^2) + h_n(1-x(2-x)) \]
\[ = h_n(1-x^2) + h_n(1-2x+x^2) \]
\[ = h_n((1-x)(1+x)) + h_n((1-x)^2) \]
\[ = h_n((1-x)(2-(1-x))) + h_n((1-x)^2) \]
\[ = h_{n+1}(1-x). \]

**Problem 8.**
We know that
\[ P(X = 1) = P(X = 1 \land Y = 0) + P(X = 1 \land Y = 1) = \frac{1}{2} \]
and hence \( P(X = 0) = \frac{1}{2} \), as well. On the other hand
\[ P(Y = 1) = P(X = 1 \land Y = 0) + P(X = 0 \land Y = 0) = \frac{1}{2} \]
and hence \( P(Y = 0) = \frac{1}{2} \). We now obtain
\[ P(X = 0)P(Y = 0) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \neq \frac{1}{2} = P(X = 0 \land Y = 0). \]
Hence, \( X \) and \( Y \) are not independent.