Mock Final Exam
This mock exam is very similar to the final exam given last year. The exam will neither be graded, nor corrected. You can try to solve it at home within 3 hours. The more you adhere to the time limits, the more realistic the exam will be. Make sure not to use anything for it (no books, scripts, calculators, etc).

The exam has 100 points, distributed over five multiple choice problems and three problems where you are asked to compute a solution or write a proof. For each of the multiple-choice questions there is exactly one correct answer.

Problem 1. [5 × 2 points] For each of the following sets, mark the correct answer:
A) finite  B) infinite and countable  C) uncountable

   i) \( \mathbb{R} \times \{\emptyset\} \)
   ii) \( \mathbb{R} \times \mathbb{Q} \)
   iii) \( \mathbb{Q} \times \mathbb{Q} \times \mathbb{Z} \)
   iv) \( \{m \in \mathbb{Z} \mid m^2 \equiv 2 \mod 4\} \)
   v) The set of all even prime numbers.
Problem 2. [4 × 3 points] Pick any sequence \( f_0, f_1, f_2, f_3, \ldots \) and let \( F(x) \) be the generating function corresponding to this sequence. For each of the following generating functions \( G(x) \), what is the corresponding sequence?

i) \( G(x) = F(x^2) \)

ii) \( G(x) = xF(x) \)

iii) \( G(x) = F'(x) \), where \( F'(x) \) denotes the derivative of \( F(x) \)

iv) \( G(x) = \frac{F(x) + F(-x)}{2} \)

A) \( 0, 0, f_0, f_1, f_2, f_3, \ldots \)

B) \( f_0, 0, f_1, 0, f_2, 0, f_3, \ldots \)

C) \( f_0, f_0 + f_1, f_0 + f_1 + f_2, f_0 + f_1 + f_2 + f_3, \ldots \)

D) \( f'_0, f'_1, f'_2, f'_3, \ldots \)

E) \( \frac{1}{f_0}, \frac{f_0}{f_1}, \frac{f_1}{f_2}, \ldots \)

F) \( 0, f_1, 2f_2, 3f_3, \ldots \)

G) \( f_0^2, f_1^2, f_2^2, f_3^2, \ldots \)

H) \( \frac{1}{f_0}, -\frac{f_1}{f_0}, \frac{f_1 - f_0f_2}{f_0}, \ldots \)

I) \( \frac{1}{f_0}, -\frac{f_1}{f_0}, \frac{f_2 - f_0f_2}{f_0}, \ldots \)

J) \( f_0, 0, f_2, 0, f_4, \ldots \)

K) \( xf_0, xf_1, xf_2, xf_3, \ldots \)

L) \( \frac{1}{f_0}, \frac{1}{f_1}, \frac{1}{f_2}, \ldots \)

M) \( 0, f_0, f_1, f_2, f_3, \ldots \)

N) \( 1, f_1, 2, f_2, 3, f_3, \ldots \)

O) \( f_1, 2f_2, 3f_3, \ldots \)
Problem 3. [2 × 4 pts]

i) The remainder of the division of $11^7$ by 13 is equal to

ii) The remainder of the division of $3^{170}$ by 13 is equal to

A) 0  
B) 1  
C) 2  
D) 3  
E) 4  
F) 5  
G) 6  
H) 7  
I) 8  
J) 9  
K) 10  
L) 11  
M) 12
Problem 4. [10 points] There are \( n \) pigeons, where \( n \in \mathbb{N}_{\geq 1} \). Out of these \( n \) pigeons, \( k \) are smart and know about the pigeonhole principle, where \( k < n/2 \). The remaining \( n - k \) pigeons are not-so-smart. Each smart pigeon eats at least 6 worms and at most 10 worms, and each not-so-smart pigeon eats at most 4 worms (but might also eat no worm and remain hungry!). At least how many pigeons eat the same number of worms? (We are looking for the best lower bound.)

A) \( \lfloor \frac{n-k}{4} \rfloor \)
B) \( \lfloor \frac{k}{4} \rfloor \)
C) \( \lceil \frac{n-k}{5} \rceil \)
D) \( \lceil \frac{k}{5} \rceil \)
E) \( \lceil \frac{k(n-k)}{5} \rceil \)
Problem 5. [8 points] In the following, $X$ and $Y$ are random variables defined on the same sample space $S$, and $E$ and $F$ are events. For each of the following statements indicate whether it is true (T) or false (F).

3. $E[XY] = E[X] + E[Y]$ if $X$ and $Y$ are independent;
4. The mapping $X : S \to \mathbb{R}$ must be surjective;
5. The mapping $X : S \to \mathbb{R}$ must be injective;
6. The mapping $X : S \to \mathbb{R}$ must be bijective;
7. The mean of the sum of $n$ Bernoulli trials with success probability $q$ is $nq(1 - q)$;
8. The mean of the sum of $n$ Bernoulli trials with success probability $q$ is $nq$. 

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Problem 6. [12 points] Determine the number of integers in \{1, 2, 3, \ldots, 2015\} that are divisible by 6, 7, or 8.
Problem 7. [20 points] Let \( h_0(x) = x(1-x) \) and define \( h_n(x) = h_{n-1}(x^2) + h_{n-1}(x(2-x)), \ n \in \mathbb{N}_{\geq 1} \). Prove that \( h_n(x) = h_n(1-x) \) for \( n \in \mathbb{N}_{\geq 0} \) and \( x \in [0, 1] \).
Problem 8. [20 points] Let $X$ and $Y$ be random variables taking values in $\{0, 1\}$. Let $p(X = 0 \land Y = 0) = p(X = 1 \land Y = 1) = 1/2$ and $p(X = 0 \land Y = 1) = p(X = 1 \land Y = 0) = 0$. Prove or disprove that $X$ and $Y$ are independent.

NOTE: $p(X = x \land Y = y)$ denotes the probability of the event that the random variable $X$ takes on the value $x$ AND the random variable $Y$ takes on the value $y$. 