Problem 1.

1. We use strong induction to show the statement.
   
   **Basis step:** We can form $8 = 3 + 5$, $9 = 3 + 3 + 3$ and $10 = 5 + 5$ cents with just 3 and 5 cents stamps.

   **Induction step** $(P(n-3) \land P(n-2) \land P(n-1)) \rightarrow P(n)$: We want to form $n$ cents for $n \geq 11$. We know by the induction hypothesis that we can form the amount of $n-3$ cents with 3 and 5 cents stamps. Thus, we can add a 3 cents stamp to form $n$. \(\square\)

2. We use mathematical induction to show the statement.
   
   **Basis step:** For $n = 1$ we have a $2 \times 2$ square. If we remove any one square, we have an $L$-shape, i.e., the obtained shape is also $L$-coverable.

   **Induction step** $(n-1) \rightarrow n$: Let $n \geq 2$. The induction hypothesis says that any shape in $C_{n-1}$ is $L$-coverable. Consider the $2^n \times 2^n$ grid and divide it into 4 grids of size $2^{n-1} \times 2^{n-1}$:

   ![Diagram](image1)

   For any shape in $C_n$ the removed square, let us call it $s_1$, is in one of the smaller grids.

   ![Diagram](image2)

   By the induction hypothesis this smaller grid without the removed square is in $C_{n-1}$ and is hence $L$-coverable. For the other three grids we remove the square closest to the center of the large grid:
Again, by induction hypothesis, we know that those three grids without the removed squares are $L$-coverable. Since the respective three squares form an $L$-shape, we get that the big grid without $s_1$, i.e., the shape in $C_n$, is $L$-coverable.

Problem 2.

1. We get $f(0) = 0, f(1) = -1, f(2) = -2$ etc. In general, we get the formula $f(n) = -n$. We will now prove this formula by induction.

   Basis step: For $n = 0$, we get $f(0) = -0 = 0$, i.e., the formula holds.

   Induction step $n \rightarrow n + 1$: As induction hypothesis we assume that $f(n) = -n$.

   Then we get
   $$f(n + 1) = f(n) - 1 \overset{IH}{=} -n - 1 = -(n + 1).$$

   Thus, the formula holds for $n + 1$.

2. We get $f(0) = 0, f(1) = 1, f(2) = 0, f(3) = 2, f(4) = 0, f(5) = 4$ etc. In general we get the formula

   $$f(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 2^{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases}.$$ 

   We will now prove this formula by induction.

   Basis step: For $n = 0$ we get $f(0) = 0$, for $n = 1$ we get $f(1) = 1 = 2^0$, i.e., the formula holds for $n = 0, 1$.

   Induction step $(P(n - 1) \land P(n)) \rightarrow P(n + 1)$: As induction hypothesis we assume that the formula holds for $f(n)$ and $f(n - 1)$. Then we get

   $$f(n + 1) = 2f(n - 1) \overset{IH}{=} 2 \left\{ \begin{array}{ll} 0 & \text{if } n-1 \text{ is even} \\ 2^{\frac{n-2}{2}} & \text{if } n-1 \text{ is odd} \end{array} \right\} = \left\{ \begin{array}{ll} 0 & \text{if } n+1 \text{ is even} \\ 2^{\frac{n}{2}} & \text{if } n+1 \text{ is odd} \end{array} \right\},$$

   since $n + 1$ is odd if and only if $n - 1$ is odd. Thus, the formula holds for $n + 1$.

Problem 3.

1. We compute $a_1 = 2 + 1 = 3$ and

   $$a_n - a_{n-1} = 2n + 1 - (2(n - 1) + 1) = 2.$$ 

   Hence we get that $a_n = a_{n-1} + 2$, which defines the sequence, together with $a_1 = 3$.
2. We compute \( a_1 = 10 \) and 
\[
\frac{a_n}{a_{n-1}} = \frac{10^n}{10^{n-1}} = 10.
\]
Hence we get that \( a_n = 10a_{n-1} \), which defines the sequence, together with \( a_1 = 10 \).

3. We compute \( a_1 = 2, a_2 = 6 \) and 
\[
a_n - a_{n-1} = n(n + 1) - (n - 1)n = n(n + 1 - (n - 1)) = 2n.
\]
Hence we get that \( a_n = a_{n-1} + 2n \). To get rid of the \( 2n \) in this formula we notice that \( a_{n-1} - a_{n-2} = 2(n - 1) \), i.e., \( 2n = a_{n-1} - a_{n-2} + 2 \). Thus, we get that 
\[
a_n = a_{n-1} + (a_{n-1} - a_{n-2} + 2) = 2a_{n-1} - a_{n-2} + 2
\]
defines the sequence, together with \( a_1 = 2 \) and \( a_2 = 6 \).

**Problem 4.** We prove the two statements by mathematical induction.

1. Basis step: For \( n = 1 \) we get \( f_1^2 = 1^2 = 1 \) and \( f_1f_2 = 1 \), hence the statement is true for \( n = 1 \).

   Induction step \( n \to n + 1 \): We assume that the statement is true for \( n \). Then, for \( n + 1 \) we get
   
   \[
   \begin{align*}
   f_1^2 + f_2^2 + \cdots + f_n^2 + f_{n+1}^2 \\
   &=(f_1^2 + f_2^2 + \cdots + f_n^2) + f_{n+1}^2 \\
   &\overset{IH}=f_nf_{n+1} + f_{n+1}^2 \\
   &=(f_n + f_{n+1})f_{n+1} \\
   &=f_{n+2}f_{n+1}
   \end{align*}
   \]

2. Basis step: For \( n = 1 \) we get \( f_2^2 = 1^2 = 1 \) and \( f_0f_1 + f_1f_2 = 0 + 1 = 1 \), hence the statement is true for \( n = 1 \).

   Induction step \( n \to n + 1 \): We assume that the statement is true for \( n \). Then, for \( n + 1 \) we get
   
   \[
   \begin{align*}
   f_0f_1 + f_1f_2 + \cdots + f_{2(n+1)-1}f_{2(n+1)} \\
   &=(f_0f_1 + f_1f_2 + \cdots + f_{2n-1}f_{2n}) + f_{2n}f_{2n+1} + f_{2n+1}f_{2n+2} \\
   &\overset{IH}=f_{2n}f_{2n+1} + f_{2n+1}f_{2n+2} \\
   &=f_{2n}(f_{2n} + f_{2n+1}) + f_{2n+1}f_{2n+2} \\
   &=f_{2n}f_{2n+2} + f_{2n+1}f_{2n+2} \\
   &=(f_{2n} + f_{2n+1})f_{2n+2} \\
   &=f_{2n+2}^2
   \end{align*}
   \]
Problem 5.

1. We first compute the number of passwords of length five, which is the number of all possible strings minus the number of strings that contain only letters. Since there are 26 letters and 10 digits, we get \(36^5 - 26^5\) possible passwords of length 5. Then we do the same for passwords of length six; we get \(36^6 - 26^6\) of those. To get the number of all possible passwords we need to add the two numbers, i.e., there are \(36^6 + 36^5 - 26^6 - 26^5 = 1916451360\) possible passwords.

2. There are \(1000000/4 = 250000\) integers less than 1000001 that are divisible by 4. Similarly, there are \([1000000/6] = 166666\) integers less than 1000001 that are divisible by 6. For the inclusion-exclusion principle we also need to count the integers that are divisible by 4 and by 6. The integers that are divisible by 6 and by 4 are exactly the integers that are divisible by 12 (since 12 is the least common multiple of 4 and 6). There are \([1000000/12] = 83333\) many of these less than 1000001. Hence we get

\[
250000 + 166666 - 83333 = 333333
\]

integers less than 1000001, that are divisible by 4 or 6. Thus, there are \(1000000 - 333333 = 666667\) integers less than 1000001, that are not divisible by 4 or 6.

Problem 6.

1. If the string has exactly four 1s then it has six 0s. Hence, we need to count how many possible ways there are to put the four 1s, then the rest is automatically filled with 0s. Since we have 10 coordinates, there are \(\binom{10}{4} = \frac{10!}{6!4!} = 210\) possible ways of putting the 1s, and therefore 210 bit strings of length 10 with four 1s.

2. To count all strings with at most four 1s, we can add the number of strings with four, three, two, one and no 1s, respectively. Similarly to 1), the separate numbers are given by \(\binom{10}{4}, \binom{10}{3}, \binom{10}{2}, \binom{10}{1}\) and \(\binom{10}{0}\). Hence, overall there are

\[
\binom{10}{4} + \binom{10}{3} + \binom{10}{2} + \binom{10}{1} + \binom{10}{0} = 210 + 120 + 45 + 10 + 1 = 386
\]

bit strings of length 10 with at most four 1s.

3. To count the number of bit strings with at least four 1s, we can also count the number of bit strings with less than four 1s and subtract it from the number of all bit strings. Similarly to 2), there are

\[
\binom{10}{3} + \binom{10}{2} + \binom{10}{1} + \binom{10}{0} = 120 + 45 + 10 + 1 = 176
\]

bit strings with at most three 1s. Hence there are \(2^{10} - 176 = 1024 - 176 = 848\) bit strings with at least four 1s.

4. If there are equally many 0s and 1s, there must be exactly five 0s and five 1s. Thus, we count the number of bit strings with exactly five 1s, which is \(\binom{10}{5} = 252\).