Exercise 1: Exponential STDP Window

1. Since there is only a single presynaptic spike we can assume that initially $x_j^{pre} = 0$, $y_i^{post} = 0$. Integrating $x_j^{pre}$ gives

$$
\tau_+ x_j^{pre} = \Theta(t - t_j^{pre}) \int -x_j^{pre} + \delta(t - t_j^{pre}) dt
$$

Now we use this expression in the weight update equation and integrate again

$$
\Delta w_{ij} = a_+(w_{ij}) \int_{-\infty}^{\infty} x_j^{pre} \delta(t - t_i^{post}) dt
$$

$$
= a_+(w_{ij}) \int_{t_j^{pre}}^{\infty} \exp(-\frac{t - t_j^{pre}}{\tau_+}) \delta(t - t_i^{post}) dt
$$

$$
= a_+(w_{ij}) \exp(-\frac{t_i^{post} - t_j^{pre}}{\tau_+})
$$

2. In analogy to above we get a weight update

$$
\Delta w_{ij} = a_-(w_{ij}) \exp(-\frac{t_i^{post} - t_j^{pre}}{\tau_-})
$$

Exercise 2: STDP and Hebbian Rate Model

1. The temporal average is defined as follows

$$
\langle f(t) \rangle_T = \frac{1}{T} \int_{t}^{t+T} f(s) ds
$$

We define further $S_i(t) = \sum_k \delta(t - t_i^k)$ the spike train emitted by neuron $i$, where the sum runs over all firing times $t_i^k$. We are interested in the averaged weight change per unit time during the interval $T$.

$$
\left\langle \frac{d}{dt} w_{ij}(t) \right\rangle_T = \int_{-\infty}^{\infty} W(s) \langle S_i(t-s) S_j(t) \rangle_T ds
$$
\[= \nu_i \nu_j \int_{-\infty}^{\infty} W(s) ds\]

where we used

\[\langle S_i(t-s)S_j(t) \rangle_T = \nu_i \nu_j\]

We therefore see the Hebbian nature of the learning rule. Here the integral over the learning window \(W\) corresponds to \(a_2^{corr}\).

2. In a more realistic scenario presynaptic spikes are the cause of postsynaptic spiking i.e. the two spike trains are correlated.

We define the response kernel \(\epsilon(x)\) is the response kernel \(\epsilon(x) = \mu \Theta(x) \Theta(\Delta - x) S_j(x)\) with the Heaviside function \(\Theta\) and the time window \(\Delta\) that describes the effect of a presynaptic spike on the postsynaptic neuron. We also define \(\mu = 10Hz\).

Temporal averaging now gives

\[
\left< \frac{d}{dt} w_{ij} \right>_T = \nu_i \nu_j \int_{-\infty}^{\infty} W(s) ds + \int_{-\infty}^{\infty} \langle W(s)\epsilon(t-s) \rangle_T ds
\]

\[= \nu_i \nu_j \int_{-\infty}^{\infty} W(s) ds + \mu \nu_j \int_{-\infty}^{\infty} W(s) \Theta(s) \Theta(\Delta - s) ds
\]

\[= \nu_i \nu_j \int_{-\infty}^{\infty} W(s) ds + \mu \nu_j \int_{0}^{\Delta} W(s) ds
\]