Problem 1. Prove that for any five points selected inside an equilateral triangle with side length equal to 1, there always exists a pair whose distance is $\leq 1/2$.

Problem 2. Let $a_i$ for $i \geq 0$ be an arithmetic progression with initial term $a$ and common difference $d$, and let $g_i$ for $i \geq 0$ be a geometric progression with initial term $g$ and common ratio $r$. To avoid trouble we assume $a > 0, d > 0, g > 0, r > 0$, and $r \neq 1$. In class we have seen the closed formulas $\sum_{i=0}^{k} a_i = (k+1)(a + \frac{dk}{2})$ and $\sum_{i=0}^{k} g_i = \frac{g(r^{k+1} - 1)}{r - 1}$ for the summations of arithmetic and geometric progressions. Find closed formulas for the following summations:

1. $\sum_{i=1}^{k} (a_i - a_{i-1})$
2. $\sum_{i=1}^{k} (g_i - g_{i-1})$
3. $\sum_{i=0}^{k} a_i g_i$
4. $\sum_{i=0}^{k} \frac{a_i}{g_i}$

Problem 3. For the integers $i$ with $1 \leq i < 6$ let $M_i$ be an $s_i \times s_{i+1}$ matrix over the real numbers. Furthermore, let

$$P_3 = M_1 \cdot M_2 \cdot M_3$$
$$P_4 = M_1 \cdot M_2 \cdot M_3 \cdot M_4$$
$$P_5 = M_1 \cdot M_2 \cdot M_3 \cdot M_4 \cdot M_5.$$ 

In this exercise we consider how to compute $P_j$ given the $M_i$-matrices, where we assume that we use traditional matrix multiplication, requiring $kmn$ multiplications of real numbers (plus additions, which we do not count) to multiply a $k \times m$ matrix and an $m \times n$ matrix.

1. Show for $j = 3, 4, 5$ that $P_j$ is a well-defined $s_1 \times s_{j+1}$ matrix over the real numbers.

2. To compute $P_3 = M_1 \cdot M_2 \cdot M_3$, we can first compute $M_{1,2} = M_1 \cdot M_2$ and then calculate $P_3$ as $M_{1,2} \cdot M_3$; or we can first compute $M_{2,3} = M_2 \cdot M_3$ and then
calculate $P_3$ as $M_1 \cdot M_{2,3}$. We refer to these two different ways of computing $P_3$ as two different ways to \textit{parenthesize} the expression $M_1 \cdot M_2 \cdot M_3$, namely as

$$(M_1 \cdot M_2) \cdot M_3$$

for the first way to compute $P_3$ and

$$M_1 \cdot (M_2 \cdot M_3)$$

for the second way. Counting multiplications in $\mathbb{R}$, which of the two ways is the “best” (smallest total number of multiplications), if $s_i = s_{i-1} + 1$ for $i \geq 2$?

3. Assuming we want to calculate just $P_4$ (so, we are not interested in $P_3$), in how many different ways can the expression for $P_4$ be parenthesized? Which one is the best, and which one is the worst, if $s_1 = 3, s_2 = 6, s_3 = 4, s_4 = 1, s_5 = 2, s_6 = 7$ for $i \geq 2$?

4. Assuming we want to calculate just $P_5$ (so, we are not interested in $P_3$ or $P_4$), in how many different ways can the expression for $P_5$ be parenthesized?

\textbf{Problem 4.} The following two algorithms sort the input sequence $a_0, a_1, \ldots, a_n$ of real numbers in ascending order:

\begin{align*}
\textbf{Algorithm 1 Selection Sort} \\
&\text{for } i = 0 \text{ to } n - 1 \text{ do} \\
&\quad min := i + 1 \\
&\quad \text{for } j = i + 1 \text{ to } n \text{ do} \\
&\quad \quad \text{if } a_{min} > a_j \text{ then} \\
&\quad \quad \quad min := j \\
&\quad \text{end if} \\
&\text{end for} \\
&\text{if } a_i > a_{\text{min}} \text{ then} \\
&\quad \text{swap } a_i \text{ and } a_{\text{min}} \\
&\text{end if} \\
&\text{end for}
\end{align*}

\begin{align*}
\textbf{Algorithm 2 Insertion Sort} \\
&\text{for } j = 1 \text{ to } n \text{ do} \\
&\quad i := 0 \\
&\quad \text{while } a_j > a_i \text{ do} \\
&\quad \quad i := i + 1 \\
&\quad \text{end while} \\
&\quad m := a_j \\
&\quad \text{for } k = 0 \text{ to } j - i - 1 \text{ do} \\
&\quad \quad a_{j-k} := a_{j-k-1} \\
&\quad \text{end for} \\
&\quad a_i := m \\
&\text{end for}
\end{align*}

1. Use Selection Sort to sort the sequence

$$9, 12, -43, 20, -2, 3, 7, 28, 19$$

and write down each step of the algorithm. Then, do the same with Insertion Sort.

2. What is the approximate overall cost of the two algorithms for an input sequence of length $n + 1$?

\textbf{Problem 5.} Let $f(n) = a_k n^k + a_{k-1} n^{k-1} + a_{k-2} n^{k-2} + \cdots + a_1 n + a_0$ for some positive integer $k$ and real numbers $a_0, a_1, \ldots, a_k$. Show that $f(n)$ is $O(n^k)$. (\text{Hint}: Use the triangle inequality $|a + b| \leq |a| + |b|$.)

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