Exercise 1: Separation of time scales

A. One-dimensional system

Consider the following differential equation
\[ \tau \frac{dx}{dt} = -x + c. \]  

1.1 Find the fixed point \( x_0 \) of this system. Hint: a fixed point is a stationary solution \( \Rightarrow \frac{dx}{dt} = 0 \).

1.2 Show that the fixed point is a stable one, and that the solution of (1) converges exponentially towards the fixed point with a time constant \( \tau \). Hint: write down the solution assuming an initial condition \( x(t = 0) \neq x_0 \).

1.3 Consider the case where \( c \) is time-dependent, namely,
\[ c \equiv c(t) = \begin{cases} 
0 & \text{for } t < 0 \\
c_0 & \text{for } 0 \leq t < 1 \\
0 & \text{for } t > 1.
\end{cases} \]
Calculate the solution \( x(t) \) with initial condition \( x(t = -10) = 0 \).

1.4 Take the expression \( x(t) \) you have found in the previous question. Consider \( \tau = 0.5 \) and \( \tau = 0.01 \) and sketch the function graph.

B. Separation of time scales

Consider the following system of equations:
\[ \frac{du}{dt} = f(u) - m \]
\[ \epsilon \frac{dm}{dt} = -m + c(u) \]
with \( \epsilon = 0.01 \).

1.5 Exploit the fact that \( \epsilon \ll 1 \) and reduce the system to one equation (note the similarity between the \( m \)-equation and Eq.(1)).

1.6 Set \( f(u) = -au + b \) where \( a > 0, b \in \mathbb{R} \) and \( c(u) = \tanh(u) \). Discuss the stability of the fixed points with respect to \( a \) and \( b \). Hint: use the graphical analysis for one dimensional equations from week 1: when plotting \( f(u) \) and \( c(u) \) against \( u \), you can read off the fixed point from that graph.
Exercise 2: Phase plane stability analysis

2.1 Linear system

Consider the following linear system:
\[
\begin{align*}
\frac{du}{dt} &= \alpha u - w \\
\frac{dw}{dt} &= \beta u - w.
\end{align*}
\]

These equations can be written in matrix form as \( \frac{d}{dt} x = Ax \) where \( x = \begin{pmatrix} v \\ w \end{pmatrix} \) and \( A = \begin{pmatrix} \alpha & -1 \\ \beta & -1 \end{pmatrix} \). Determine the conditions for stability of the point \((u = 0, w = 0)\) in the case \( \beta > \alpha \) by studying the eigenvalues of the above matrix. (Hint: Distinguish the cases of real and complex eigenvalues.)

2.2 Piecewise linear Fitzhugh-Nagumo model

The Fitzhugh-Nagumo model is defined by the equations
\[
\begin{align*}
\frac{du}{dt} &= F(u, w) = f(u) - w + I \\
\frac{dw}{dt} &= G(u, w) = bu - w
\end{align*}
\]

Here, \( u(t) \) is the membrane potential and \( w(t) \) is a second, time-dependent variable. \( I \) stands for the injected current. A simplified model is obtained by considering a piecewise linear \( f(u) \):
\[
f(u) = \begin{cases} 
-u & \text{if } u < 1 \\
\frac{u-1}{a} - 1 & \text{if } 1 \leq u < 1 + 2a \\
2(1 + a) - u & \text{if } u > 1 + 2a
\end{cases}
\]

with \( a < 1, b > 1/a \).

(i) Sketch the “nullclines” \( \frac{du}{dt} = 0 \) and \( \frac{dw}{dt} = 0 \) in a \((u,v)\)-plot. Consider the case \( I = 0 \). How does the fixed point move as \( I \) is varied? Sketch the form of the flow (i.e., the vector \((\frac{du}{dt}, \frac{dw}{dt})\)) along the nullclines and deduce qualitatively the shape of the trajectories.

(ii) Calculate the Jacobian matrix evaluated at the fixed point,
\[
J = \begin{pmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial w} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial w} \end{pmatrix}.
\]

Determine, by studying the eigenvalues of \( J \), the linear stability of the fixed point as a function of \( I \). What happens when the fixed point destabilizes?